Gromov-Hausdorff distance and applications

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Summer school

Metric Geometry

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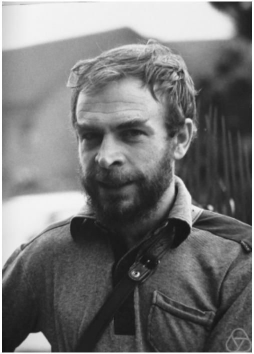
Felix Hausdorff, 1868–1942


Mikhail Gromov *1943

Sources

- Burago, Burago, Ivanov: A course in metric geometry
- Bridson, Haefliger: Metric spaces of non-positive curvature
- Heinonen: Geometric embeddings of metric spaces
- Gromov: Groups of polynomial growth and expanding maps
- Petersen: Riemannian geometry and several articles
- Hausdorff: Set theory
- Kuratowski: Topology

Metrics and pseudometrics

- A pseudometric on a set X is a function $d : X \times X \to [0; \infty)$ with $d(x,x) = 0$, $d(x,y) = d(y,x)$, and triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z)$$

- A metric is a pseudometric such that $d(x,y) = 0$ only when $x = y$.

Remarks

- For a pseudometric space (X,d),

$$xRy \iff d(x,y) = 0$$

  is an equivalence relation. The pseudometric d induces a metric on the quotient X/R.
- If d is a pseudometric and $\delta > 0$, then $d_\delta(x,y) = d(x,y) + \delta$ (for $x \neq y$) defines a metric.
- Often useful to admit $d : X \times X \to [0; \infty]$. Then $d(x,y) < \infty$ is an equivalence relation.

Complete, compact, separable

A metric space (X, d) is called

- complete if every Cauchy sequence converges
- compact if every sequence has a convergent subsequence
- separable if there is countable dense subset
- totally bounded (= precompact) if $\forall\, \varepsilon > 0\ \exists$ a finite ε-dense subset $X_\varepsilon \subseteq X$, i.e.
$$X = \bigcup_{x \in X_\varepsilon} B_\varepsilon(x)$$

Implications

- separable $\Leftrightarrow$ topology has a countable basis
- totally bounded $\Rightarrow$ separable
- totally bounded $\Leftrightarrow$ every sequence has a Cauchy subsequence
- compact $\Leftrightarrow$ complete and totally bounded

Examples: classical sequence spaces

Examples

- Closed balls in ℓ^2 are complete, separable, not compact.
- Closed balls in ℓ^∞ are complete, not separable.

Recall definitions: For $1 \leq p \leq \infty$, ℓ^p is the space of sequences

$$x = (x_k)_{k\in\mathbb{N}} = (x_1, x_2, \dots)$$

of real numbers such that the ℓ^p-norm

$$||x||_p = \begin{cases} \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p} & \text{for } p < \infty \\ \sup_{k\in\mathbb{N}} |x_k| & \text{for } p = \infty \end{cases}$$

is finite. Banach spaces, for $p=2$ Hilbert. For $p \leq q$,

$$\ell^1 \subset \ell^p \subset \ell^q \subset c_0 \subset c \subset \ell^\infty$$

Fréchet embedding

Theorem. Every separable metric space (X, d) admits an isometric embedding into ℓ^∞.

Proof. Choose a dense sequence $(x_k)_{k\in\mathbb{N}}$ in X and define $\phi : X \to \ell^\infty$ by

$$\phi(x) = \big(\phi_k(x)\big)_{k\in\mathbb{N}} = \big(d(x, x_k) - d(x_k, x_0)\big)_{k\in\mathbb{N}}$$

Then for $x, y \in X$,

$$|\phi_k(x) - \phi_k(y)| = |d(x, x_k) - d(y, x_k)| \leq d(x, y)$$

with equality obtained when x_k approaches x or y. Therefore,

$$||\phi(x) - \phi(y)||_\infty = d(x, y)\ . \quad \square$$

Exercise. Find a metric space (X, d) consisting of four points that does *not* admit an isometric imbedding into Hilbert space ℓ^2.

Cauchy completion and precompactness

Theorem. For every metric space (X, d) there is a *complete* metric space $(\hat{X}, \hat{d})$ with an isometric embedding $\iota\colon X \to \hat{X}$ such that $\iota(X)$ is dense in $\hat{X}$.

- $(\hat{X}, \hat{d})$ unique up to isometry, called the completion of (X, d).
- Construction: Generalize Cantor's definition of the real numbers from the rationals. Define a pseudometric on the set of all Cauchy-sequences in X by
$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) := \lim_{k\to\infty} d(x_k, y_k)\,.$$
Then define $\hat{X}$ to be the quotient metric space identifying elements with distance zero. Thus points of $\hat{X}$ are equivalence classes
$$\xi = [(x_1, x_2, \dots)]$$
of Cauchy sequences in X, where equivalence means having distance zero.

Theorem. (X, d) precompact $\iff$ $(\hat{X}, \hat{d})$ compact.

Hausdorff distance

For a subset $A \subseteq X$ of a metric space (X, d), the r-neighbourhood of A is defined as

$$U_r(A) := \{x \in X \mid \operatorname{dist}(x, A) < r\} = \bigcup_{x \in A} B_r(x)$$

Hausdorff-distance of non-empty subsets $A, B \subseteq X$:

$$\begin{aligned} d_H(A, B) &:= \inf\{r > 0 \mid A \subseteq U_r(B) \text{ and } B \subseteq U_r(A)\} \\ &= \max\left\{ \sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A) \right\} \end{aligned}$$

Properties

- d_H satisfies triangle inequality, is a pseudometric on the set of bounded subsets of X.
- $d_H(A, B) = d_H(A, \overline{B})$
- $d_H(A, B) = 0 \iff \overline{A} = \overline{B}$
- $d_H(\{a\}, \{b\}) = d(a, b)$

Hausdorff compactness theorem

Let $\mathfrak{C}(X)$ be the set of all *non-empty closed bounded* subsets of X, equipped with the metric d_H.

Theorem

- If X is complete, then $\mathfrak{C}(X)$ is complete (Hahn 1932).
- If X is totally bounded, then $\mathfrak{C}(X)$ is totally bounded.
- If X is compact, then $\mathfrak{C}(X)$ is compact (Hausdorff, Blaschke).

Remark

- Same if $\mathfrak{C}(X)$ denotes the set of all non-empty *compact* subsets.

History

Blaschke selection theorem (1916). Every d_H-bounded sequence of compact convex sets $A_k \subseteq \mathbb{R}^n$ subconverges to a compact convex set $A \subseteq \mathbb{R}^n$.

- Proof. There is a compact $X \subset \mathbb{R}^n$ that contains every A_k. Apply previous theorem to obtain a subsequence converging to some $A \in \mathfrak{C}(X)$ and check that compact Hausdorff-limits of convex sets are convex.

Hausdorff compactness: proof

- $\mathfrak{C}(X)$ is totally bounded: Given $\varepsilon > 0$, choose a finite subset $X_\varepsilon \subseteq X$ that is ε-dense in X. Then the power set $\mathfrak{P}(X_\varepsilon)$ is an ε-dense finite subset of $\mathfrak{C}(X)$.

- $\mathfrak{C}(X)$ is complete: Let $A_k \in \mathfrak{C}(X)$ be a Cauchy sequence. Define

$$A := F\limsup A_k = \bigcap_{n=1}^{\infty} \overline{A_n \cup A_{n+1} \cup \dots}$$

Claim. $A \in \mathfrak{C}(X)$, and $d_H(A_n, A) \to 0$ as $n \to \infty$.

- Remark : The set $F\limsup A_k$ is called the upper closed limit of the sequence $(A_k)_{k\in\mathbb{N}}$. An alternative description is

$$\begin{aligned} F\limsup A_k &= \{x \in X \mid \forall \varepsilon > 0 : B_\varepsilon(x) \cap A_k \neq \emptyset \text{ for } \infty \text{ many } k\} \\ &= \{\text{accumulation points of sequences } a_n \in A_n\} \end{aligned}$$

Proof of claim

- $A \subseteq U_\varepsilon(A_n)$ for all large n : $a \in A$ implies that the $\varepsilon/2$-ball around a meets infinitely many of the A_k, and so
$$a \in U_{\varepsilon/2}(A_k)$$
for these k. Since the sequence is Cauchy, we have $A_k \subseteq U_{\varepsilon/2}(A_n)$ for all large k and n. Hence
$$a \in U_{\varepsilon/2}(U_{\varepsilon/2}(A_n)) \subseteq U_\varepsilon(A_n)\,.$$
- $A_n \subseteq U_\varepsilon(A)$ for all large n : If $x \in A_n$ for sufficiently large n, then there is a subsequence $n = n_1 < n_2 < \dots$ and a sequence of points $a_{n_k} \in A_{n_k}$ starting at $a_1 = x$ such that $d(a_{n_k}, a_{n_{k+1}}) < \varepsilon/2^{k+1}$. The sequence $(a_{n_k})_{k\in\mathbb{N}}$ is Cauchy, hence converges to some $a \in X$, and by definition of A we have $a \in A$. By the triangle inequality,
$$d(x,a) \le \sum_{k=1}^{\infty} d(a_{n_k}, a_{n_{k+1}}) < \varepsilon. \quad \square$$

Description of Hausdorff limits: topological limits

Recall the upper closed limit of the sequence $(A_k)_{k\in\mathbb{N}}$:

- $F\lim\sup A_k = \{x \in X \mid \forall\varepsilon > 0 : B_\varepsilon(x) \cap A_k \neq \emptyset \text{ for } \infty \text{ many } k\}$
- The lower closed limit is defined as
  $F\lim\inf A_k = \{x \in X \mid \forall\varepsilon > 0 : B_\varepsilon(x) \cap A_k \neq \emptyset \text{ for nearly all } k\}$
- The closed limit is said to exist if both are equal :

$$F\lim A_k := F\lim\inf A_k = F\lim\sup A_k \,.$$

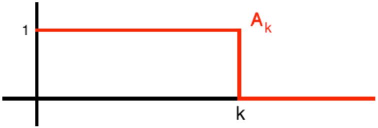
Theorem

Consider $A_k, A \in \mathfrak{C}(X)$.

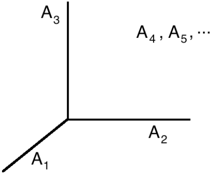
- If $d_H(A_k, A) \to 0$, then $F\lim A_k$ exists and is equal to A.
- If X is compact and $F\lim A_k$ exists, then $d_H(A_k, A) \to 0$.

Examples

- (X, d) euclidean plane,
  $F\lim A_k$ exists, but sequence
  not Hausdorff convergent:

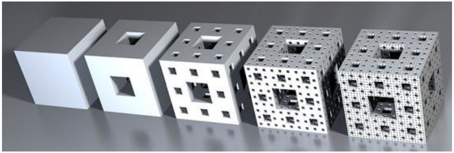


- (X, d) unit ball in Hilbert space ℓ^2,
  $F\lim A_k$ exists, but sequence not
  Hausdorff convergent:



- Menger sponge

  Source: Wikipedia



Gromov-Hausdorff distance

Definition. The Gromov-Hausdorff distance between metric spaces X and Y is defined as

$$d_{GH}(X,Y) = \inf_{Z} \inf_{X',Y'} d_H^Z(X',Y')$$

where the infimum $\in [0,\infty]$ is taken over all metric spaces Z and all subspaces X', Y' of Z that are isometric to X, Y.

Comments

- d_H^Z denotes the Hausdorff distance in the metric space (Z, d^Z).
- X', Y' carry the metrics obtained by restriction of d^Z.
- Distance depends only on the *isometry classes* of X and Y.
- Reformulate:

  $$d_{GH}(X,Y) = \inf_{Z} \inf_{\phi,\psi} d_H^Z(\phi(X),\psi(Y))$$

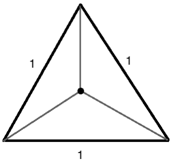
  where the infimum is over all isometric embeddings $\phi : X \to Z$ and $\psi : Y \to Z$.

Examples

- If X_ε is an ε-dense subset of X with the induced metric, then $d_{GH}(X, X_\varepsilon) < \varepsilon$. So totally bounded metric spaces admit approximation by finite metric spaces.
- If $\{p\}$ is a one-point space, then $d_{GH}(X, \{p\}) = \frac{1}{2}\operatorname{diam}(X)$, where

$$\operatorname{diam}(X) = \sup_{x,y \in X} d(x, y)$$

Proof: Take $Z = X \sqcup \{p\}$ and extend the given metric from X to Z by setting $d(x, p) := \frac{1}{2}\operatorname{diam}(X)$.



- If diameters are finite, then

$$\frac{1}{2}\left|\operatorname{diam}(X) - \operatorname{diam}(Y)\right| \leq d_{GH}(X, Y) \leq \frac{1}{2}\max\{\operatorname{diam}(X), \operatorname{diam}(Y)\}.$$

Lipschitz-close implies GH-close

A map $F : X \to Y$ between metric spaces is called L-bi-lipschitz if

$$\frac{1}{L}\, d(x,x') \le d(Fx,Fx') \le L\, d(x,x')\,.$$

Claim. If $F : X \to Y$ is $(1+\varepsilon)$-bi-lipschitz and bijective, then

$$d_{GH}(X,Y) \le \varepsilon \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}\,.$$

Proof. Can assume diameters are finite. Take $Z = X \sqcup Y$ and extend the metrics d^X and d^Y to a metric(!) on Z by setting

$$d(x,y) := \inf_{a\in X} \left(d^X(x,a) + d^Y(y,Fa)\right) + \varepsilon C$$

where $C = \max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$. Given $y \in Y$, we show that $x = F^{-1}y \in X$ has distance at most εC from y: For every $a \in X$

$$d(x,y) \le d^X(F^{-1}y, a) + d^Y(y, Fa) + \varepsilon C\,.$$

Choose $a = F^{-1}y$ to obtain $d(x,y) \le \varepsilon C$. □

Alternative definition 1

Proposition

$$d_{GH}(X,Y) = \inf_d d_H^{X \sqcup Y}(X,Y)$$

where $X \sqcup Y$ is the disjoint union and the infimum is taken over all admissible metrics d on $X \sqcup Y$, i.e. metrics that extend d^X and d^Y.

Proof

- If $\widehat{d_{GH}}(X,Y)$ denotes the right hand side, then $d_{GH} \leq \widehat{d_{GH}}$ because the infimum for d_{GH} is extended over a larger set.
- Conversely given $\varepsilon > 0$, choose Z, X' and Y' such that

$$d_H^Z(X',Y') \leq d_{GH}(X,Y) + \varepsilon\,. \qquad (*)$$

- If X', Y' disjoint, restrict the metric of Z to the union $X' \cup Y'$ to get

$$\widehat{d_{GH}}(X,Y) = \widehat{d_{GH}}(X',Y') \leq d_{GH}(X,Y) + \varepsilon\,. \qquad (**)$$

- If X', Y' are *not* disjoint, replace Z, X', Y' by $Z \times [0,1]$, $X' \times \{0\}$, $Y' \times \{\varepsilon\}$. Obtain equations $(*)$ and $(**)$ with ε replaced by 2ε. □

d_{GH} is a metric

Claim

$$d_{GH}(X,Z) \leq d_{GH}(X,Y) + d_{GH}(Y,Z)$$

Proof. Take admissible metrics $d^{X \sqcup Y}$ and $d^{Y \sqcup Z}$ and, for $\delta > 0$, define an admissible metric $d^{X \sqcup Z}$ on the disjoint union $X \sqcup Z$ by

$$d^{X \sqcup Z}(x,z) = \inf_{y \in Y} \left(d^{X \sqcup Y}(x,y) + d^{Y \sqcup Z}(y,z)\right) + \delta$$

for $x \in X$ and $z \in Z$. Then

$$d_H^{X \sqcup Z}(X,Z) \leq d_H^{X \sqcup Y}(X,Y) + d_H^{Y \sqcup Z}(Y,Z) + \delta\,.$$

Now take the infimum over all admissible metrics $d^{X \sqcup Y}$ and $d^{Y \sqcup Z}$, and finally let $\delta \to 0$. □

Proposition. X, Y compact with $d_{GH}(X,Y) = 0$, then X, Y are isometric.

Notation. Let $\mathfrak{M}$ denote the set of isometry classes of compact metric spaces $\neq \emptyset$. Then $(\mathfrak{M}, d_{GH})$ is a metric space.

d_{GH} is a metric

Proof of proposition. Take a sequence of admissible metrics d_k on $X \sqcup Y$ such that the Hausdorff distance between X and Y with respect to d_k is $\leq 1/k$. Then there are (discontinuous) maps $I_k : X \to Y$ and $J_k : Y \to X$ with

$$d_k(x, I_k(x)) \leq \tfrac{1}{k} \quad \text{and} \quad d_k(y, J_k(y)) \leq \tfrac{1}{k} .$$

The triangle inequality for d_k then implies

$$\begin{aligned}
d(I_k(x_1), I_k(x_2)) &\leq \tfrac{2}{k} + d(x_1, x_2) \\
d(J_k(y_1), J_k(y_2)) &\leq \tfrac{2}{k} + d(y_1, y_2) \\
d(x, J_k \circ I_k(x)) &\leq \tfrac{2}{k} \\
d(y, I_k \circ J_k(y)) &\leq \tfrac{2}{k}
\end{aligned}$$

An Arzela-Ascoli argument yields limits $I : X \to Y$ and $J : Y \to X$ for $k \to \infty$. (Obtain $I : X \to Y$ first on a countable dense subset $A \subseteq X$ using a diagonal argument and the compactness of Y, then extend from A to X. Similarly for J.) Then I and J are the required isometries. □

Counterexample

Example. Two proper metric spaces with $d_{GH}(X, Y) = 0$ that are not isometric.

- Both X and Y are metric graphs obtained from the real line by attaching segments of suitable length at all integer points.

- For X attach a segment of length $|\sin(m)|$ to the point $m \in \mathbb{Z}$.
- For Y attach a segment of length $|\sin(m + \frac{1}{2})|$ to the point $m \in \mathbb{Z}$.

- To see that $d_{GH}(X, Y) \leq \varepsilon$ for every $\varepsilon > 0$, observe that X and Y are isometrically embedded into the grid

$$Z = \{(x, y) \in \mathbb{R}^2 \mid x \text{ or } y \in \mathbb{Z}\}$$

equipped with its path metric. A suitable integer translation in the x-direction will move X into an ε-neighborhood of Y.

Alternative definition 2

Proposition. For separable metric spaces X, Y,

$$d_{GH}(X, Y) = \inf_{\phi,\psi} d_H^\infty(\phi(X), \psi(Y))$$

where the infimum is taken over all isometric embeddings $\phi : X \to \ell^\infty$ and $\psi : Y \to \ell^\infty$, and d_H^∞ is the Hausdorff distance in ℓ^∞.

Proof. The inequality $\leq$ is clear. Conversely given $\varepsilon > 0$, choose an admissible metric d on $Z = X \sqcup Y$ such that

$$d_H^Z(X, Y) \leq d_{GH}(X, Y) + \varepsilon\,.$$

Since (Z, d) is also separable, there is an isometric embedding $\iota : Z \to \ell^\infty$, and we obtain isometric embeddings

$$\phi : X \to X \sqcup Y \to \ell^\infty \qquad \psi : Y \to X \sqcup Y \to \ell^\infty\,.$$

Then

$$d_H^\infty(\phi(X), \psi(Y)) = d_H^Z(X, Y) \leq d_{GH}(X, Y) + \varepsilon\,. \quad \square$$

Correspondences

Definition. Consider metric spaces X and Y.

- A correspondence (or *surjective relation*) between X and Y is a subset
$$\mathcal{R} \subseteq X \times Y$$
such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ remain surjective when restricted to $\mathcal{R}$.
  - Example: If $f : X \to Y$ is a surjective map, then the graph $\mathcal{R} = \{(x, f(x)) \mid x \in X\}$ is a correspondence.
- The distortion of a correspondence is defined as
$$\mathrm{dis}(\mathcal{R}) = \sup_{(x,y),(x',y') \in \mathcal{R}} |d^Y(y, y') - d^X(x, x')|$$
  - Remark: If $\mathrm{dis}(\mathcal{R}) = 0$, then $\mathcal{R}$ is the graph of an isometry.

Alternative definition 3

Theorem

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R})$$

where the infimum is taken over all correspondences $\mathcal{R} \subseteq X \times Y$.

Proof

- $d_{GH}(X,Y) \geq \frac{1}{2} \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R})$:

  If $r > d_{GH}(X,Y)$, then there is a metric space (Z,d) containing X and Y such that the Hausdorff distance in Z satisfies $d_H(X,Y) < r$. Then

  $$\mathcal{R} := \{(x,y) \mid d(x,y) < r\}$$

  is a correspondence, and

  $$\frac{1}{2} \operatorname{dis}(\mathcal{R}) < r$$

  because for $(x,y),(x',y') \in \mathcal{R}$

  $$|d(y,y') - d(x,x')| \leq d(x,y) + d(x',y') < 2r\,.$$

Alternative definition 3

- $d_{GH}(X, Y) \leq \frac{1}{2} \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R})$:

  Let $\mathcal{R}$ be a correspondence and $r := \frac{1}{2}\operatorname{dis}(\mathcal{R})$. We may assume $r > 0$. Define an admissible metric(!) on $Z = X \sqcup Y$ by

  $$d(x, y) = \inf_{(x', y') \in \mathcal{R}} \left(d(x, x') + r + d(y', y)\right).$$

  Then the Hausdorff distance of $X, Y \subseteq Z$ is

  $$d_H(X, Y) \leq r = \frac{1}{2}\operatorname{dis}(\mathcal{R}) \ :$$

  Given $x \in X$, choose $y \in Y$ such that $(x, y) \in \mathcal{R}$. Then

  $$d(x, y) \leq d(x, x) + r + d(y, y) = r\,,$$

  and so the distance from x to Y is $\leq r$. □

ε-isometries

Definition. A map $f : X \to Y$ is called an ε-isometry if its distortion

$$\mathrm{dis}(f) := \sup_{x,x' \in X} |d^Y(fx, fx') - d^X(x, x')| \leq \varepsilon$$

and if $f(X)$ is ε-dense in Y.

Proposition

- If $d_{GH}(X, Y) < \varepsilon$, then there is a 2ε-isometry $f : X \to Y$.
- If there is an ε-isometry $f : X \to Y$, then $d_{GH}(X, Y) \leq \frac{3}{2}\varepsilon$.

Proof. Use the previous theorem. If $d_{GH}(X, Y) < \varepsilon$, take a correspondence with $\mathrm{dis}(\mathcal{R}) < 2\varepsilon$. For each x choose y such that $(x, y) \in \mathcal{R}$ and define $f(x) = y$. Then f is a 2ε-isometry.

Given an ε-isometry $f : X \to Y$, define $\mathcal{R} := \{(x, y) \mid d(fx, y) < \varepsilon\}$. This is a correspondence with $\mathrm{dis}(\mathcal{R}) \leq 3\varepsilon$. $\square$

GH-limits

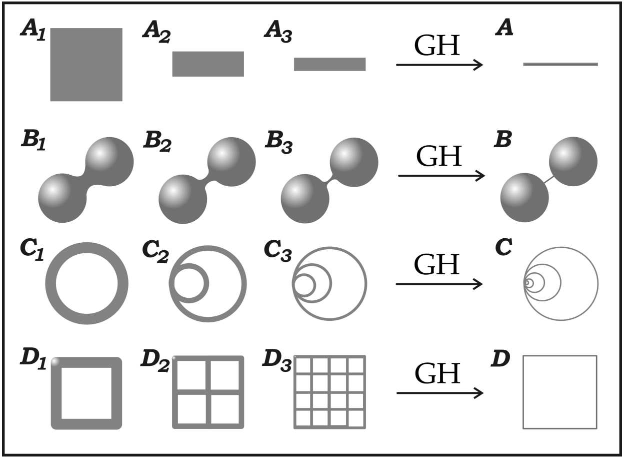
Definition. A sequence of metric spaces X_k converges to X in the Gromov-Hausdorff sense (short: GH-converges to X) if $d_{GH}(X_k, X) \to 0$ as $k \to \infty$. Notation:

$$X_k \xrightarrow{GH} X \quad (k \to \infty)$$

Remarks

- If X is compact, then X is unique up to isometry.
- Example: Every compact X is a GH-limit of a sequence of finite metric spaces.
- Hausdorff convergent implies GH-convergent.
- Assume X_k, X compact, and $X_k \xrightarrow{GH} X$. Then there are $X'_k, X' \subseteq \ell^\infty$ isometric to X_k, X such that $X'_k \xrightarrow{d_H^\infty} X'$.
  Proof later.

GH-convergence: pictures



Source: Christina Sormani, How Riemannian manifolds converge

Examples: bounded curvature collaps

- Circles; flat tori; $M \times S^1$
- The Hopf fibration $$S^3 \xrightarrow{h} \mathbb{C}P^1 = S^2$$
  is the quotient map of the free isometric S^1-action
  $$e^{i\theta}(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$$
  on the standard sphere $S^3 \subseteq \mathbb{C}^2$. This is a Riemannian submersion for a metric of constant curvature $=4$ on $\mathbb{C}P^1$.
  - Take cyclic groups $C_k \subseteq S^1$ of order k. Then
    $$S^3/C_k \xrightarrow{GH} \mathbb{C}P^1 \quad \text{as } k \to \infty$$
  - Berger spheres: Define $S^3_\varepsilon = (S^3, g_\varepsilon)$, where the Riemannian metric g_ε is obtained by multiplying the standard Riemannian metric of S^3 with a factor $\varepsilon > 0$ in the fiber direction. Then
    $$S^3_\varepsilon \xrightarrow{GH} \mathbb{C}P^1 \quad \text{as } \varepsilon \to 0\,.$$

Examples: Heisenberg group

The 3-dimensional Heisenberg group $\mathbb{H}$ is the set of all

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } x, y, z \in \mathbb{R}\,.$$

The subset $\Gamma \subseteq \mathbb{H}$ of integral matrices is a discrete subgroup. Consider the compact manifold $M = \Gamma\backslash\mathbb{H}$.

1. For $\varepsilon > 0$, take basis for left invariant 1-forms

   $$\omega^1 = \varepsilon\, dx \qquad \omega^2 = \varepsilon\, dy \qquad \omega^3 = \varepsilon^3(dz - x dy)$$

   Define Riemannian metric so that this is an ON-basis:

   $$g_\varepsilon = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$$

   This is left invariant, and $g_\varepsilon \to 0$ as $\varepsilon \to 0$. Conclusion:

   $$(M, g_\varepsilon) \xrightarrow{GH} \text{point} \qquad \text{as } \varepsilon \to 0\,.$$

Examples: Heisenberg (continued)

- The curvature in this example remains bounded: The Maurer-Cartan equations $d\omega^k = c^k_{ij}\omega^i \wedge \omega^j$ are

$$d\omega^1 = d\omega^2 = 0 \qquad d\omega^3 = -\varepsilon\,\omega^1 \wedge \omega^2$$

For the curvature tensor R one then calculates $||R|| \leq 6||d\omega|| = 6\varepsilon$. So curvature $\to 0$ as $\varepsilon \to 0$.

  - This works for general nilpotent Lie groups G: choose basis for $\mathfrak{g}^*$ such that $c^k_{ij} = 0$ unless $i, j < k$. These metrics descend to nilmanifold quotients $\Gamma \backslash G$; and to compact infranil-quotients $\Lambda \backslash G$ after averaging over Λ / Γ.

2. Now consider the Riemannian metrics g'_ε given by the ON-basis

$$\omega^1 = dx \qquad \omega^2 = dy \qquad \omega^3 = \tfrac{1}{\varepsilon}(dz - xdy)\,.$$

For $\varepsilon \to 0$, (M, g'_ε) converges to a metric space X which is M equipped with the subriemannian metric defined by ω^1, ω^2 on the plane field $\ker \omega^3$. Curvatures go to $\pm\infty$.

Properties inherited by GH-limits

Proposition. Suppose $X_k \xrightarrow{GH} Y$. If each X_k is/has ..., then Y is/has ...

- separable
- totally bounded
- a proper space – if Y is complete
- a length space – if Y is complete
- a proper geodesic space – if Y is complete
- diameter $\leq D$ (in fact $\operatorname{diam} X_k \to \operatorname{diam} Y$)
- properties of the form $F(d_{11}, d_{12}, \ldots, d_{k-1,k}) \geq 0$ or $= 0$, where $d_{ij} = d(x_i, x_j)$, and where F is continuous, e.g.
  - δ-hyperbolic
  - CBB$^\kappa$ ($\Leftrightarrow$ $(1+3)^\kappa$-condition)
  - CAT$^\kappa$ ($\Leftrightarrow$ $(2+2)^\kappa$-condition)
- complete geodesic with curv $\geq \kappa$
- NOT: complete geodesic with curv $\leq \kappa$ (counterexample: hyperboloids $\to$ double-cone)

Proofs: totally bounded

- For CBB^κ, CAT^κ and curv see the lectures of Stephanie Alexander at this summer school.
- Totally bounded. Pick a finite ε-dense subset in some X_k GH-close to Y, then move it to Y via a correspondence. Details:
  - Given $\varepsilon > 0$, fix k so large that $d_{GH}(X_k, Y) < \varepsilon/4$. Then there is a correspondence $\mathcal{R} \subseteq X_k \times Y$ with distortion $\mathrm{dis}(\mathcal{R}) < \varepsilon/2$. Take a finite $\varepsilon/2$-dense subset $X_k' \subset X_k$. For each $x' \in X_k'$ choose a $y' \in Y$ such that $(x', y') \in \mathcal{R}$, and let Y' be the set of all such y'.
  - We claim that Y' is ε-dense in Y:
  - Given $y \in Y$, find $x \in X_k$ such that $(x, y) \in \mathcal{R}$, and then $x' \in X_k'$ at distance $< \varepsilon/2$ from x. For the $y' \in Y'$ that corresponds to this x' we obtain

$$\begin{aligned} d(y, y') &\leq |d(y, y') - d(x, x')| + d(x, x') \\ &\leq \mathrm{dis}(\mathcal{R}) + d(x, x') \\ &< \varepsilon\,. \end{aligned}$$

□

Proofs: proper

- Proper. A metric space X is called proper if all closed balls

$$\bar{B}_r(x) := \{d(\cdot, x) \leq r\}$$

  are compact.

  - Given a ball $\bar{B}_r(y) \subseteq Y$, there are $x_k \in X_k$ corresponding to y, and then for radii $r_k \searrow r$ the balls $\bar{B}_{r_k}(x_k)$ Hausdorff-converge to $\bar{B}_r(y)$.
  - Since all $\bar{B}_{r_k}(x_k)$ are totally bounded, so is the limit $\bar{B}_r(y)$. Since Y is complete, $\bar{B}_r(y)$ is complete, hence compact. $\square$

Proofs: length space

- Length space. A length space is a metric space X such that $d(x, x')$ is the infimal length of curves joining x and x'. Recall the approximate mid point condition:

  For all $x, x' \in X$ and $\varepsilon > 0$, there is an ε-midpoint $m \in X$, i.e.

$$\max\{d(x,m), d(m,x')\} \leq \tfrac{1}{2} d(x,x') + \varepsilon\,.$$

  Then
  - length space $\Longrightarrow$ approximate mid point condition
  - $\Longleftarrow$ is true for *complete* metric spaces

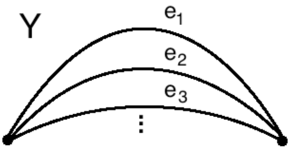
  - Verify this condition for Y: Given $y, y' \in Y$ and $\varepsilon > 0$, fix k so large that there is a correspondence with small distortion between X_k and Y. Take points $x_k, x_k' \in X_k$ corresponding to y, y' and find an ε-midpoint $m_k \in X_k$. Finally, let $m \in Y$ be a point corresponding to m_k. Then m is a 3ε-midpoint for y, y'. Since Y is assumed complete, it is a length space. □

Proofs: proper geodesic

- Proper and geodesic. By definition, a geodesic space is a length space such all pairs x, x' can be joined by a curve of length $= d(x, x')$. So every X_k is a proper length space, and so is the limit Y. By the Hopf-Rinow-theorem, every proper length space is geodesic. $\square$

- Example. A complete limit Y of geodesic spaces X_k that is not geodesic: Y is the metric graph constructed by joining two vertices with a sequence of edges e_n of length $1+\frac{1}{n}$ for $n=1,2,\ldots$

  X_k is obtained from Y by replacing the edge e_k by an edge of length 1.



Packing and covering

Definition. For a metric space X and $\varepsilon > 0$ define the covering and packing numbers by

$$\mathrm{cov}(X,\varepsilon) = \min\{\, n \mid X \text{ can be covered by } n \text{ closed } \varepsilon\text{-balls}\}$$
$$\mathrm{pack}(X,\varepsilon) = \sup\{\, n \mid X \text{ contains } n \text{ disjoint } \tfrac{\varepsilon}{2}\text{-balls}\}.$$

Lemma 1. $\mathrm{cov}(X,\varepsilon) \leq \mathrm{pack}(X,\varepsilon)$.

Proof. If $B_{\varepsilon/2}(x_1), \ldots, B_{\varepsilon/2}(x_n)$ is a maximal disjoint set of $\varepsilon/2$-balls, then the balls $\bar{B}_\varepsilon(x_1), \ldots, \bar{B}_\varepsilon(x_n)$ cover X. □

Lemma 2. If $d_{GH}(X, Y) \leq \delta$, then

$$\mathrm{cov}(X,\varepsilon) \geq \mathrm{cov}(Y,\varepsilon + 2\delta)$$
$$\mathrm{pack}(X,\varepsilon) \geq \mathrm{pack}(Y,\varepsilon + 2\delta)$$

Proof. Use a correspondence with distortion $2\delta'$, $\delta' > \delta$. □

Totally bounded sets in $\mathfrak{M}$

Theorem. For a subset $\mathcal{C} \subseteq \mathfrak{M}$, the following are equivalent:

(1) There is a constant $D > 0$ and a function $N : (0, \infty) \to \mathbb{N}$ such that $\operatorname{diam}(X) \leq D$ and $\operatorname{pack}(X, \varepsilon) \leq N(\varepsilon)$ for all $X \in \mathcal{C}$.

(2) Same as (1), but replace $\operatorname{pack}(X, \varepsilon)$ by $\operatorname{cov}(X, \varepsilon)$.

(3) $\mathcal{C}$ is totally bounded with respect to d_{GH}.

Proof

(3)⇒(1) Recall that (3) means $\forall \delta > 0$ $\exists$ finite δ-dense subset in $\mathcal{C}$. Consider such a subset $\mathcal{C}' \subseteq \mathcal{C}$ and let D' and $N'(\varepsilon)$ be upper bounds for $\operatorname{diam}(\cdot)$ and $\operatorname{pack}(\cdot, \varepsilon)$ on $\mathcal{C}'$.

Given $X \in \mathcal{C}$, take $C \in \mathcal{C}'$ such that $d_{GH}(X, C) < \delta$. Then

$$\operatorname{diam}(X) \leq \operatorname{diam}(C) + 2\delta \leq D' + 2\delta$$
$$\operatorname{pack}(X, \varepsilon) \leq \operatorname{pack}(C, \varepsilon - 2\delta) \leq N'(\varepsilon - 2\delta).$$

(1)⇒(2) by Lemma 1.

Totally bounded sets in $\mathfrak{M}$

(2)⇒(3) Fix $\varepsilon > 0$.

- The set $\mathcal{F}$ of finite metric spaces with at most $N(\varepsilon)$ elements and diameters $\leq D$ is totally bounded with respect to d_{GH}.

  Proof. With each $F \in \mathcal{F}$ that has $N \leq N(\varepsilon)$ elements associate "the" $N \times N$ matrix $\Delta(F) = (d_{ij})$ of pairwise distances of all the points in F. These matrices have entries bounded by D, so they form a totally bounded set in $\mathbb{R}^{N \times N}$. If $\Delta(F)$ and $\Delta(F')$ are δ-close, then there is a correspondence (in fact a bijection) between F and F' with distortion $< \delta$, and so $d_{GH}(F, F') \leq \delta$.

- This set $\mathcal{F}$ is ε-dense for $\mathcal{C}$.

  Proof. Given $X \in \mathcal{C}$, cover it by $\leq N(\varepsilon)$ balls of radius ε. Let F be the set of centers of these balls. Then $F \in \mathcal{F}$, and $d_{GH}(X, F) \leq \varepsilon$.

- This works for every $\varepsilon > 0$. Conclude that every sequence in $\mathcal{C}$ contains a Cauchy subsequence (diagonal argument). □

Lemma (Gromov). For every totally bounded subset $\mathcal{C} \subseteq \mathfrak{M}$ there is a compact subset $K \subseteq \ell^\infty$ such that every $X \in \mathcal{C}$ admits an isometric embedding into K.

As a corollary we obtain:

Theorem. The metric space $(\mathfrak{M}, d_{GH})$ is complete.

Proof

Apply the Lemma to the set of terms $\{X_k \mid k \in \mathbb{N}\} \subseteq \mathfrak{M}$ of a given Cauchy sequence. The lemma says that the X_k have isometric copies X_k' contained in some compact $K \subseteq \ell^\infty$. The Hausdorff compactness theorem applied to K provides a subsequence X_{k_j}' that d_H^∞-converges to a compact $X \subseteq K$. This implies that $X_{k_j} \xrightarrow{GH} X$. Since the sequence was Cauchy, $X_k \xrightarrow{GH} X$. $\square$

The space $\ell^\infty(A)$

It remains to prove Gromov's lemma. Instead of embeddings into $\ell^\infty = \ell^\infty(\mathbb{N})$, we construct embeddings into $\ell^\infty(A)$ for some other countably infinite set A. This is the Banach space of all bounded functions $f : A \to \mathbb{R}$ with the sup-norm. It is isometric to $\ell^\infty(\mathbb{N})$.

Definition. Fix a sequence $\mathbf{N} = (N_1, N_2, \dots)$ of positive integers and consider the sets

$$A_1 = \{(n_1) \mid n_1 = 1, \dots, N_1\}$$

$$A_2 = \{(n_1, n_2) \mid n_1 = 1, \dots, N_1;\ n_2 = 1, \dots, N_2\}$$

$$A_3 = \{(n_1, n_2, n_3) \mid n_1 = 1, \dots, N_1;\ n_2 = 1, \dots, N_2;\ n_3 = 1, \dots, N_3\}$$

etc., and then

$$A = \bigcup_{j=1}^\infty A_j$$

The elements $f \in \ell^\infty(A)$ are bounded families of numbers

$$(f(a))_{a \in A} = (f_a)_{a \in A}$$

where the indices a are of the form $a = (n_1, \dots, n_k)$. We write $f(n_1, \dots, n_k)$ instead of $f\big((n_1, \dots, n_k)\big)$.

Compact sets in $\ell^\infty(A)$

Sublemma. Let $D > 0$, and let $\mathbf{e} = (\varepsilon_1, \varepsilon_2, \dots)$ be a sequence of positive numbers such that $\sum_{j=1}^{\infty} \varepsilon_j < \infty$. Consider the subset $F = F_{D,\mathbf{e}} \subseteq \ell^\infty(A)$ defined by the following conditions:

(1) $0 \le f(n_1) \le D$ for $n_1 = 1, \dots, N_1$

(2) $|f(n_1, \dots, n_k, n_{k+1}) - f(n_1, \dots, n_k)| \le \varepsilon_k$

for all k and all $(n_1, \dots, n_{k+1}) \in A$. Then F is compact.

Proof

F is closed in $\ell^\infty(A)$, hence complete. Therefore it suffices to show that F is totally bounded. Note that we have finite dimensional subspaces

$$\ell^\infty(A_1 \cup \dots \cup A_k) \hookrightarrow \ell^\infty(A).$$

- $F \cap \ell^\infty(A_1 \cup \dots \cup A_k)$ is compact.
- By condition (2), F is contained in the $\hat{\varepsilon}_k$-neighbourhood of $F \cap \ell^\infty(A_1 \cup \dots \cup A_k)$, where $\hat{\varepsilon}_k = \varepsilon_k + \varepsilon_{k+1} + \dots \to 0$ as $k \to \infty$.
- Using this, every sequence in F has a Cauchy subsequence (diagonal sequence argument). So F is totally bounded. □

Proof of Gromov lemma

Recall the statement: For every totally bounded $\mathcal{C} \subseteq \mathfrak{M}$ there is a compact $K \subseteq \ell^\infty(\mathbb{N})$ such that every $X \in \mathcal{C}$ admits an isometric embedding into K.

Proof

- Choose $D > 0$ and a function $N : (0, \infty) \to \mathbb{N}$ such that $\operatorname{diam}(X) \leq D$ and $\operatorname{cov}(X, \varepsilon) \leq N(\varepsilon)$ for all $X \in \mathcal{C}$.
- Take a decreasing sequence $\mathbf{e} = (\varepsilon_1, \varepsilon_2, \dots)$ of positive numbers such that $\sum_{j=1}^\infty \varepsilon_j < \infty$, and let $N_j := N(\varepsilon_j)$.
- Using this sequence $N_1, N_2, \dots,$ define A as before, and let

$$K := F_{D,2\mathbf{e}} \subseteq \ell^\infty(A) \cong \ell^\infty(\mathbb{N})$$

be the compact set described in the sublemma. We show that every $X \in \mathcal{C}$ embeds isometrically into this K.

Proof of Gromov lemma (end)

- Cover X with N_1 balls of radius ε_1, say $B(x_{n_1}, \varepsilon_1)$ where $n_1 = 1, \ldots, N_1$.

  Next cover each of the balls $B(x_{n_1}, \varepsilon_1)$ with N_2 balls of radius ε_2, say $B(x_{n_1 n_2}, \varepsilon_2)$ where $n_2 = 1, \ldots, N_2$.

  Then cover each of these balls $B(x_{n_1 n_2}, \varepsilon_2)$ with N_3 balls of radius ε_3, say $B(x_{n_1 n_2 n_3}, \varepsilon_3)$ where $n_3 = 1, \ldots, N_3$. Continue like this.

- The centers x_a, $a \in A$ of all these balls form a dense set in X. Therefore the Fréchet-embedding $\phi : X \to \ell^\infty(A)$ defined by

$$\phi(x) = \big(\phi_a(x)\big)_{a \in A} = \big(d(x, x_a)\big)_{a \in A}$$

  is isometric.

- Verify that $\phi(X) \subseteq F_{D,2\mathbf{e}}$: Condition (1) holds since $d(x, x_{n_1}) \leq D$, and condition (2) because of

$$|d(x, x_{n_1 \ldots n_k n_{k+1}}) - d(x, x_{n_1 \ldots n_k})| \leq d(x_{n_1 \ldots n_k n_{k+1}}, x_{n_1 \ldots n_k}) \leq 2\varepsilon_k \quad \square$$

Topics

- For non-compact spaces: pointed GH-convergence
- What Gromov does with it: groups of polynomial growth
- Precompact sets of Riemannian manifolds: the Bishop-Gromov relative volume comparison
- If suitable X and Y are GH-close, then X and Y are diffeomorphic, homeomorphic, homotopy equivalent; corresponding finiteness results; Cheeger, Grove, Petersen, Anderson, Perelman et.al.
- Continuity of quantities under GH-limit; Anderson's estimate on the harmonic radius of a Riemannian manifold
- Collapsing and fibration theorems: Y fixed, X close to Y, then X fibers over Y with infranil fiber; Gromov, Fukaya, Yamaguchi
- Structure of limit spaces of Riemannian manifolds under curvature bounds; Fukaya, Cheeger, Colding et.al.