Summer School: Les Diablerets 2013
Gromov hyperbolic spaces

Notes to the talks of Prof. V. Schroeder

Notes by T. Berner

Contents

Introduction

The following pages are the notes I took attending the mini-course in “Gromov hyperbolic spaces” given by Prof. V. Schroeder during the summer school 2013 in Les Diablerets.

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1 Basic Definitions and Motivation

1.1. Basic Definitions

Notations:

- (X, d) metric space
- $|xy| := d(x, y)$.

Definition 1.1 $f : X \to X'$ is

- *isometric*, if
$$|f(x)f(y)| = |xy| \quad \forall x, y \in X.$$
- *homothetic*, if $\exists \lambda \geq 0$ such that
$$|f(x)f(y)| = \lambda |xy| \quad \forall x, y \in X.$$
- A *geodesic* in X is a homothetic map $\gamma : I \to X$, where $I \subset \mathbb{R}$ some interval (where $(I, d_e(x, y) = |x - y|)$).
- X is *geodesic*, if $\forall x, y \in X\ \exists \gamma : [0, 1] \to X$ geodesic with $\gamma(0) = x$ and $\gamma(1) = y$.
  Also $\gamma([0, 1])$ is also called a *geodesic*. We denote this (not necessarily unique) geodesic with xy.

1.2. Quasi-Isometries

Motivated by geometric group theory (Cayley, Dehn).

Definition 1.2 Let G a finitely generated group, and $S \subset G$ a finite set of generators closed under inversion, that is $S = S^{-1}$.
Define the *Cayley-graph* $\Gamma(G, S)$ by

- the set of vertices is G
- there is an edge between g and g' iff $g'g^{-1} \in S$.

On $\Gamma(G, S)$ we can define a metric d_S called *word-metric* by $d_S(g, g')$ is the length of the minimal edge path between g and g'. This metric can be extended to the whole graph by identifying each edge with the interval $[0, 1]$.
This makes $(\Gamma(G, S), d_S)$ to a geodesic metric space.

Figure 1.1: *Cayley graph of* $\mathbb{Z}$ *with* $S = \{\pm 1\}$.

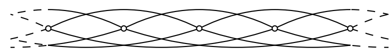

Figure 1.2: *Cayley graph of* $\mathbb{Z}$ *with* $S = \{\pm 2, \pm 3\}$.

Example 1.3 Consider $(\mathbb{Z}, \{\pm 1\})$ and $(\mathbb{Z}, \{\pm 2, \pm 3\})$

Example 1.4 $(\mathbb{Z}, \{\pm 2, \pm 3\})$

Let G a finitely generated group and consider the two generating sets S and S'. Let $a \geq 1$ such that

$$\|h\|_{S'} \leq a \quad \text{and} \quad \|h'\|_S \leq a$$

where $h, h' \in G$.
Then the identity map

$$\text{id} : (G, d_S) \to (G, d_{S'})$$

is bi-lipschitz, that is

$$\frac{1}{a}|gg'|_{S'} \leq |gg'|_S \leq a|gg'|_{S'}$$

Definition 1.5 $f : X \to X'$ is *quasi-isometric* if $\exists a \geq 1, b \geq 0$ such that $\forall x, y$

$$\frac{1}{a}|xy| - b \leq |f(x)f(y)| \leq a|xy| + b.$$

$f : X \to X'$ is called a *quasi-isometry*, if f is quasi-isometric and $\exists c \geq 0$ such that $d_H(f(x), X') \leq c$, where d_H is the Hausdorff-distance. Quasi-isometry is an equivalence relation.

Corollary 1.6 So, two Cayley-graphs $\Gamma(G, S)$ and $\Gamma(G, S')$ are quasi-isometric.

1.3. Stability of quasi-geodesics

Definition 1.7 A *quasi-geodesic* in X is a quasi-isometric map $\gamma : I \to X$. γ is called an (a, b)*-quasi-geodesic* where a, b are the bounding-constants.

Example 1.8 Consider the logarithmic spiral given by $\gamma : (0, \infty) \to \mathbb{R}^2$, where $\gamma(t) := t \cdot (\cos(\ln t), \sin(\ln t))$.
γ is a quasi-geodesic, because $\|\gamma(t)\| = t$, $\|\gamma'(t)\| = \sqrt{2}$.
So

$$\frac{1}{\sqrt{2}}|\gamma(t)\gamma(s)| \leq |t - s| \leq |\gamma(t)\gamma(s)|.$$

Theorem 1.9 (Morse) *Let* $a \geq 1$, $b \geq 0$, *then there exists* $H := H(a, b) \geq 0$ *such that for every* (a, b)*-quasi-geodesic,* $f : [0, \ell] \to \mathbb{H}^2$ *the Hausdorff-distance* $d_H(f([0, \ell]), c) \leq H$ *where* c *is a geodesic from* $f(0)$ *to* $f(\ell)$.

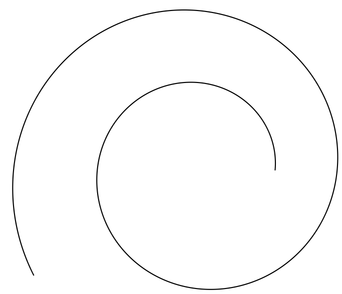

Figure 1.3: *The logarithmic spiral*

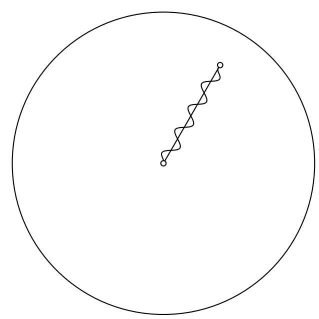

Figure 1.4: *Stability of quasi-geodesics*

Lemma 1.10 $f : [\alpha, \beta] \to \mathbb{H}^2, d_c(f(\alpha)) = d_c(f(\beta)) = M'$ then $L \geq \cosh(M') \cdot d$ where $L = \text{length}(f_{[\alpha,\beta]})$ and d_c is the distance to c.

Proof $f : [0, A] \to \mathbb{H}$ quasi-geodesic, so there are a, b such that

$$\frac{1}{a}|t - t'| - b \leq |f(t)f(t')| \leq a|t - t'| + b$$

for all $t, t' \in [0, A]$.

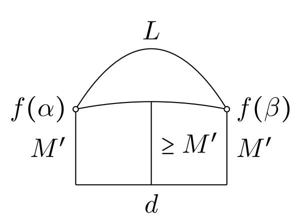

Figure 1.5

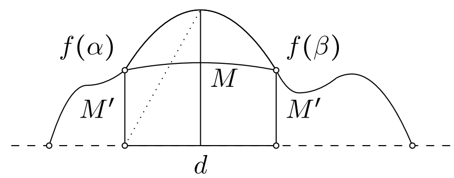

Figure 1.6

Step 1 One can assume without loss of generality that

$$\frac{1}{a}|t - t'| - b \leq |f(t)f(t')| \leq a'|t - t'| \tag{0}$$

for some $a' := a'(a, b)$.

Step 2 $M := \max d_c(f(t))$, and $M' := M/(3a^2)$.
Let $L := \text{length}(f|_{[\alpha,\beta]})$,

1. then by the previous lemma $L \geq \cosh(M') \cdot d$.
2. $L \geq 2(M - M') \geq \frac{4}{3}M \geq 4a^2M'$.
3. By (0) we have $L \leq a|\beta - \alpha| \leq a^2(|f(\alpha)f(\beta)| + b) \leq a^2(2M' + d + b)$.
4. So we have $d \geq 2M' - b$.

Assume $M' \geq b$ then by 4 we have

$$d \geq M' \geq b.$$

So $L \leq 4a^2d$ by 3.
Hence $\cosh(M') \cdot d \leq L \leq 4a^2d$.

So $M' \leq \text{Function}(a)$, and $M' \leq F(a, b)$. ■

2 Hyperbolic spaces

2.1. δ-hyperbolicity

Definition 2.1 Let X a metric space, and $\delta \geq 0$. Then X is *δ-hyperbolic* if $\forall x, y, z, w$ one has

$$|xy| + |zw| \leq \max\{|xz| + |yw|, |xw| + |yz\} + 2\delta \qquad \delta\text{-inequality}$$

X is called *Gromov-hyperbolic*, if $\exists \delta \geq 0$, such that X is δ-hyperbolic.

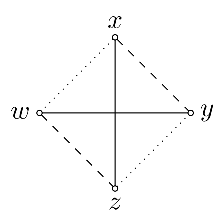

Figure 2.1: *δ-hyperbolicity*

Example 2.2 Consider a tree: The three numbers are

$$(a + c + d) + (b + c + e), \quad (a + c + e) + (b + c + d), \quad (a + b) + (d + e).$$

So this is 0-hyperbolic.

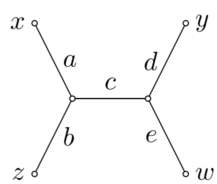

Figure 2.2: *Tree example*

Example 2.3 $\mathbb{R}^2$ is not hyperbolic: For this look at a square of side length n. The three side pairs get the numbers

$$2n, \quad 2n, \quad 2\sqrt{2}n.$$

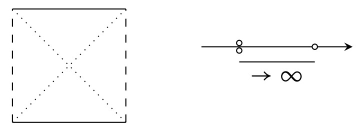

Figure 2.3: *For $n \to \infty$ the distance between the length of a side and the diagonal diverges.*

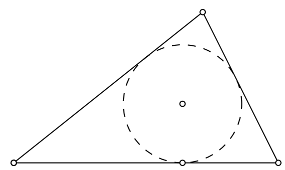

Figure 2.4: *Equiradial points and incircle*

Definition 2.4 Let X a metric space, $x, y, z, w \in X$ then we define

$$(y|z)_x := \frac{1}{2}(|xy| + xz| - |yz|).$$ Gromov product

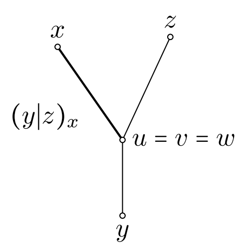

Figure 2.5: *In a tripod the equiradial points coincide*

Geometric definition (for metric spaces) incircle of the triangle.

Lemma 2.5 Let x, y, z a triangle in some metric space, then there are unique points $u \in [y, z]$, $v \in [x, z]$, $w \in [x, y]$, such that $|xw| = |xv|$, $|yw| = |yu|$ and $|zv| = |zu|$. These points are called the *equiradial points.*

Lemma 2.6 Let X a metric space, and $\delta \geq 0$. X is δ-hyperbolic, if $\forall o, x, y, z \in X$, the following holds

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\}\delta.$$

Definition 2.7 $a, b, c \in \mathbb{R}$ is called a *δ-triple* if the two smaller differ by at most δ.

Let $o, x, y, z \in X$: such that the two larger of the three numbers

$$|ox| + |yz|, \quad |oy| + |zx|, \quad |wz| + |yx|$$

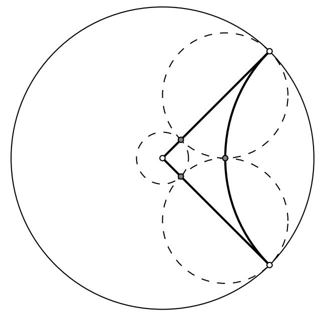

Figure 2.6: *Equiradial points for ideal triangle in* $\mathbb{H}^2$

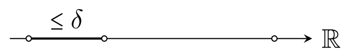

Figure 2.7: *δ-tripel*

differ by at most 2δ.
If we look now at the negative numbers

$$-|ox| - |yz|, \quad -|oy| - |zx|, \quad -|wz| - |yx|$$

the two smaller differ by 2δ.
By adding $|ox| + |oy| + |oz|$ to every entry we get

$$(|oy| + |oz| - |yz|), \quad (|ox| + |oz| - |xz|), \quad (|oy| + |ox| - |yz|)$$

which again the smaller two differ by atmost 2δ.
By multiplying these numbers with $1/2$ we get that the smallest two Gromov-products

$$(y|z)_o, \quad (x|z)_o, \quad (x|y)_o$$

differ by at most δ.

Proposition 2.8 Let X a geodesic space, then X is Gromov-hyperbolic $\Leftrightarrow \exists \delta' \geq 0$ such that if x, y, z is a triangle and u, v, w equiradial points $a \in xv$, $b \in xw$, $|xa| = |xb|$, then $|ab| \leq \delta$.

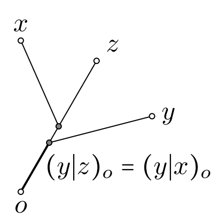

Figure 2.8

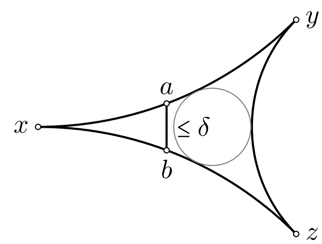

Figure 2.9: *If* $|xa| = |xb| \leq (y|z)_x$ *then* $|ab| \leq \delta$.

Proof Argument for "⇒":
Study for a geodesic $\gamma : [0, \ell] \to X$ parametrized by arclength, the distance to x: that is

$$f : \ t \mapsto |\gamma(t)x|$$

Consider $(x|y)_o$, $(x|\gamma(t))_o$, $(\gamma(t)|y)_o$. Note that $(\gamma(t)|y)_o = t$ and $(x|y)_o = \alpha$. Denote $\alpha' := (x|\gamma(t))_o$. By δ-hyperbolicity we have $|\alpha - \alpha'| \leq \delta$ ...

Theorem 2.10 (Stability of quasi-geodesics) *Let X δ-hyperbolic and geodesic. $a \geq 1$, $b \geq 0$. Then there exists $D = D(a, b)$ such that: if $\gamma : [0, \ell] \to X$ is an (a, b)-quasi-geodesic, and c is the geodesich from $\gamma(0)$ to $\gamma(\ell)$, then*

$$d_H(\gamma, c) \leq D.$$

3 Boundary at infinity

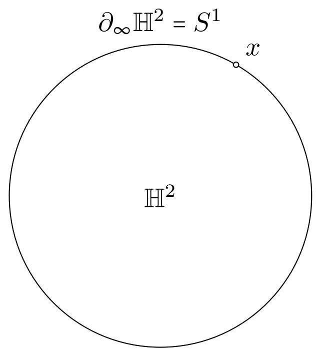

Figure 3.1

We chose a basepoint $o \in X$.

Definition 3.1 A sequence $(x_i)_{i\in\mathbb{N}}$ *converges at infinity* if

$$\lim_{i,j\to\infty}(x_i|x_j)_o = \infty.$$

Definition 3.2 If $(x_i),(y_i)$ are sequences which converge at infinity, then

$$(x_i) \sim (y_i) \quad :\Leftrightarrow \quad \lim_{i,j\to\infty}(x_i|y_i)_o = \infty$$

Lemma 3.3 $\sim$ is an equivalence relation.

Proof The problematic part is the transitivity: $(x_i) \sim (y_j) \sim (z_k) \Rightarrow (x_i) \sim (z_k)$.
$(x_i) \sim (y_j)$: $\forall K \exists N \forall i,j \geq N$: $(x_i|y_j)_o \geq K$ analgously $(y_j, z_k) \geq K$.
As $(x_i|y_j)$, $(y_j|z_k)$ and $(x_i|z_k)$ is a δ-tripel we have $(x_i|z_k) \geq K - \delta$. ∎

We define the boundary at infinity as

$$\partial_\infty X = \{[(x_i)] \mid (x_i)_{i\in\mathbb{N}} \text{ is a sequence converging at infinity}\}.$$

Let X be CAT^{-1}.

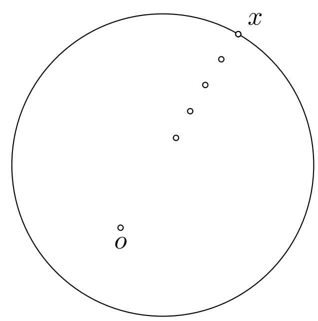

Figure 3.2

Theorem 3.5 (Bourdon) *X CAT^{-1}, $o \in X$, then*

$$\rho_o(x,y) := e^{-(x|y)_o}$$

defines a metric on $\partial_\infty X$.

Example 3.6 $\mathbb{H}^2$. Let $x, y \in \partial_\infty \mathbb{H}^2 = S^1 \in \mathbb{R}^2$, then

$$e^{-(x|y)_o} = \frac{1}{2}\|x-y\|.$$

$e^{-(x|y)_o} = \lim_{t\to\infty}(e^{h_t - 2t})^{1/2} = \sin(\vartheta/2)$

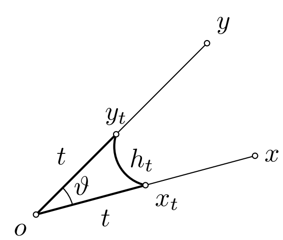

Figure 3.3

Lemma 3.7 $o, o' \in X$, then ρ_o, $\rho_{o'}$ are Möbius equivalent, i.e. they define the same crossratio.

Proof We have to show

$$\frac{\rho_o(x,y)\rho_o(z,w)}{\rho_o(x,z)\rho_o(y,w)} = \frac{\rho_{o'}(x,y)\rho_{o'}(z,w)}{\rho_{o'}(x,z)\rho_{o'}(y,w)}$$

Note that

$$\frac{\rho_o(x,y)\rho_o(z,w)}{\rho_o(x,z)\rho_o(y,w)} = \lim_{i\to\infty} e^{-1/2\cdot[(|ox_i|+|oy_i|-|x_iy_i|+|oz_i|+|ow_i|-|z_iw_i|)-(|ox_i|+|oz_i|-|x_iz_i|+|oy_i|+|ow_i|-|y_iw_i|)]}$$
$$= \lim_{i\to\infty} e^{-1/2\cdot[-|x_iy_i|-|z_iw_i|+|x_iz_i|+|y_iw_i|]}$$

So the crossratio is independent of the chosen basepoint. ■

3.1. Möbius structure

Let Z a set ($Z = \partial_\infty X$).

Definition 3.9 (extended metric) $d : Z \times Z \to [0,\infty]$, which has at most one point at infinity, that is $\exists \Omega(d) \subset Z$, $\#\Omega(d) \in \{0,1\}$, $\omega \in \Omega(d)$, $x \in \mathbb{Z} \setminus \Omega(d)$, then $d(\omega, x) = \infty$.
$Q := \{(x,y,z,w) \in \mathbb{Z}^4$, but $(x,y,x,y) \in Q$ but $(x,y,x,x) \notin Q$ (no entry may appear three times).

Definition 3.10 $\mathrm{crt}_d : Q \to \Sigma \subset \mathbb{R}P^2$

$$\mathrm{crt}_d(x,y,z,w) := (|xy||zw| : |xz||yw| : |xw||yz|).$$

$\Sigma = \{(a:b:c) \mid a,b,c$ have same sign$\}$.

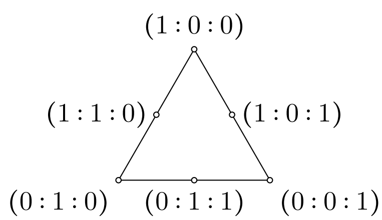

Figure 3.4: Σ

Definition 3.11 $f : X \to X'$ is *Möbius*, if

1. f injective
2. $\mathrm{crt}_{d'}(f(x),\ldots,f(w)) = \mathrm{crt}_d(x,\ldots,w)$.

Z, d, d' extended metrics on Z, d Möbius equivalent to d' iff $\mathrm{id} : (Z,d) \to (Z,d')$ is Möbius.s
A Möbius structure on Z is an equivalence class of Möbius equivalent metrics. $(Z,\mathcal{M})$ is called a Möbius space, where $\mathcal{M} = [d]$.
$\partial_\infty X$, X is CAT^{-1}, then $(\partial_\infty X, [\rho_o])$ is canonical Möbius structure on $\partial_\infty X$.
$\omega \in \partial_\infty X$

$$\rho_\omega(x,y) := \frac{\rho_o(x,y)}{\rho_o(w,x)\rho_o(w,y)}.$$

This is also a metric on $\partial_\infty X$ and $\Omega(\rho_\omega) = \{\omega\}$.
$[\rho_\omega] = [\rho_o]$ does not depend on o.

Example 3.12 $\mathbb{H}^n, \omega \in \partial_\infty \mathbb{H}^n$

- $(\partial_\infty \mathbb{H}^n, \rho_o m)$ isometric to $\mathbb{R}^{n+1} \cup \{\infty\}$.
- $(\partial_\infty \mathbb{H}^n, \rho_o)$ is isometric to $(S^{n-1}, 1/2\text{chordal metric})$.

Example 3.13 $\mathbb{CH}^2$, $-4 \le K \le -1$. $(\partial_\infty \mathbb{CH}^2, \rho_\omega)$ isometric to Heisenberggroup (Karanyi-Reinmann gauge).

Back to Gromov hyperbolic spaces.
First difficulty: define $(x|y)_o$ for points $x, y \in \partial_\infty X$.

$$\lim_{x_i \to X, y_i \to y} (x_i|y_i)_o$$

does not necessarily exist. See the following example:

Example 3.14

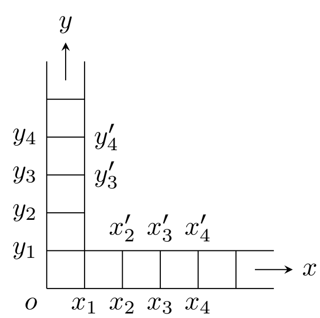

Figure 3.5: *Gromov products need not converge*

We have $(x_i|y_i)_o = 0$ and $(x'_i|y'_i)_o = 2$.

So we define it

$$(x|y)_o := \inf_{\text{sequences}} \liminf_{\substack{x_i \to x \\ y_i \to y}} (x_i|y_i)_o.$$

Then $x, y, z \in \partial_\infty X$: $(x|y)_o, (x|z)_o, (y|z)_o$ is a δ-tripel.
The second difficulty is, that $\rho_o := e^{-(x|y)_o}$ is not a metric. But it is a K-quasimetric.

Definition 3.15 ρ is a K*-quasimetric*, if

$$\rho(x,z) \le K \cdot \max\{\rho(x,y), \rho(y,z)\}.$$

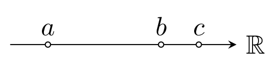

Figure 3.6: *In a K-quasimetric the quotient of the larger two numbers is boundet by K. That is $c/b \le K$*

Construction by Frink: If ρ is a K-quasimetric with $K \leq 2$, then there exists a metric $\overline{\rho}$ with

$$\frac{1}{2K}\rho \leq \overline{\rho} \leq \rho.$$

$$\overline{\rho}(x,y) := \inf_{\substack{\text{chains} \\ x=x_o,x_1,\dots,x_n=y}} \sum_{i=1}^{n} \rho(x_{i-1},x_i).$$

So we redefine

$$\rho_o := e^{-\varepsilon(x|y)_o}$$

where ε is chosen such that we can apply Frinks construction.

Definition 3.16 Let $x,y,z,w \in \partial_\infty X$. Define

$$[x,y,z,w]_o := \frac{\rho_o(x,y)\rho_o(z,w)}{\rho_o(x,z)\rho_o(y,w)}.$$

We can show, that $\exists$ constant L (depending on δ) such that

$$\frac{1}{L}[x,y,z,w]_o \leq [x,y,z,w]_{o'} \leq L[x,y,z,w]_o.$$

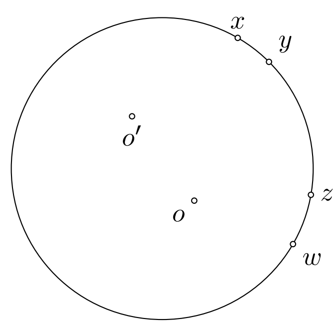

Figure 3.7

4 Morphisms

Assume we have to Gromov hyperbolic spaces X and X'.

	$F : X \to X'$	$f : \partial_\infty \to \partial_\infty X'$
	$x \mapsto x'$	
Classical Case $\mathbb{H}^n$	Isometries	Möbius maps
	Quasi-Isometries	Power-Quasi-Möbius
	$\frac{1}{a}\|xy\| - b \leq \|x'y'\| \leq a\|xy\| + b$	

Definition 4.1 f is *PQ-Möbius* (power-quasi) if $\exists p \geq 1$, $q \geq 1$, such that $\forall x, y, z, w$ with $[x, y, z, w] \geq 1$

$$\frac{1}{q}[x, y, z, w]^{1/p} \leq [x', y', z', w'] \leq q[x, y, z, w]^p.$$

Definition 4.2 Let X a metric space. $x, y, z, w \in X$. The *double-difference* is defined by

$$\begin{aligned}\langle x, y, z, w\rangle := (x|y)_o + (z|w)_o - (x|z)_o - (y|w)_o. \\ = \frac{1}{2}(|xz| + |yw| - |xy| - |zw|) \\ = (x|y)_w - (x|z)_w.\end{aligned}$$

Remark 4.3

1. $(x|y)_z = \langle x, y, z, z\rangle$
2. $|xz| = \langle x, x, y, y\rangle$

Definition 4.4 $F : X \to X$ is called *PQ-isometry* if $\exists a \geq 1, b \geq 0$, $\forall x, y, z, w$ with $\langle x, y, z, w\rangle \geq 0$

$$\frac{1}{a}\langle x, y, z, w\rangle - b \leq \langle x', y', z', w'\rangle \leq a\langle x, y, z, w\rangle + b.$$

Lemma 4.5 Let X, X' Gromov hyperbolic spaces. If $F :\to X'$ is a PQ-isometry then F extends to a map $f : \partial_\infty X \to \partial_\infty X'$ which is PQ-Möbius.

Theorem 4.6 *Let $F : X \to X'$ a quasi-isometry between geodesic spaces. Assume that X' is Gromov hyperbolic. Then*

1. *X is Gromov hyperbolic*
2. *F is a PQ-isometry*
3. *F extends to a PQ-Möbius map $\partial_\infty X \to \partial_\infty X'$.*

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