

**BOARD NOTES FOR MINICOURSE:
ALEXANDROV GEOMETRY**

These minicourse board notes are an attempt to make a few basic aspects of Alexandrov geometry easily accessible. We consider both curvature bounded above and curvature bounded below. Attention is given to examples, and to some dualities between curvature bounded above and curvature bounded below. There are many important omissions of theory, but it is hoped that this introduction will allow easier access to more advanced topics. It is hoped as well to give a hint of the beauty and power of the subject.

The notes draw on our book [AKP].

Other general sources for this subject include: [BBI]; [BH], [BS07] for curvature bounded above; [BGP92], [S93], [P102] for curvature bounded below.

Special thanks to Anton Petrunin for many interesting discussions and ideas.

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1. MODEL SPACE

- (1) (a) $\mathcal{M}(\kappa) = \mathbf{E}^2, \mathbf{S}^2(\kappa), \mathbf{H}^2(\kappa)$ (for $\kappa = 0, \kappa > 0, \kappa < 0$ respectively).
 (b) Set $\pi^\kappa = \pi/\sqrt{\kappa}$ ($= \infty$ if $\kappa \leq 0$).

Exercise 1.1. Show a closed curve of length $< 2 \cdot \pi^\kappa$ in $\mathbf{S}^2(\kappa)$ lies in an open hemisphere.

- (2) Let $(\mathcal{X}, |**| : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty])$ be a metric space.

(a) A *geodesic* γ joining x^1 and x^2 is a constant-speed path of length $|x^1 x^2|$. For a *local geodesic* γ , the parameter interval is covered by open subintervals on which γ is geodesic. We write $[x^1 x^2]$ for either γ or its image. *The choice may not be unique, but once we write $[xy]$ it means we have made a choice.*

(b) \mathcal{X} is *r-geodesic* (*r-intrinsic*) if any $x^1, x^2 \in \mathcal{X}$ with $|x^1 x^2| < r$ are joined by a geodesic (respectively, by curves of length arbitrarily close to $|x^1 x^2|$). For *geodesic* (*intrinsic*), take $r = \infty$.

(c) A *triangle* $[x^1 x^2 x^3]$ in \mathcal{X} is $[x^1 x^2] \cup [x^2 x^3] \cup [x^3 x^1]$.

(d) For $x^1, x^2, x^3 \in \mathcal{X}$, the *model triangle* $\tilde{\Delta}^\kappa[x^1 x^2 x^3]$ in $\mathcal{M}(\kappa)$ is the triangle with sidelengths $|x^1 x^2|, |x^2 x^3|, |x^3 x^1|$. The model angle $\tilde{\angle}^\kappa[x^1 x^2 x^3]$ is the angle corresponding to x^1 in $\tilde{\Delta}^\kappa[x^1 x^2 x^3]$. These are said to be *defined* if the sum of sidelengths is $< 2 \cdot \pi^\kappa$.

- (3) (a) For a *hinge* $[p_y^x] = [px] \cup [py]$ in \mathcal{X} , define the angle $\angle[p_y^x]$ by

$$\angle[p_y^x] = \lim_{\bar{x}, \bar{y} \rightarrow p} \tilde{\angle}^\kappa[p_{\bar{y}}^{\bar{x}}],$$

where $\bar{x} \in [px], \bar{y} \in [py]$, if the limit exists.

- (b) This definition of angles in \mathcal{X} is independent of κ , e.g.

$$\tilde{\angle}^\kappa[p_y^x] - \tilde{\angle}^{\bar{\kappa}}[p_y^x] \leq C(\kappa, \bar{\kappa}) \cdot |px| \cdot |py|.$$

- (c) *Triangle inequality for angles* holds:

$$\angle[p_z^x] \leq \angle[p_y^x] + \angle[p_y^z],$$

if all three angles are defined. (Use $\kappa = 0$, Euclidean cosine formula, triangle inequality for distance.)

2. DEFINITIONS

- In this column, assume \mathcal{X} complete, π^κ -geodesic,
- In this column, assume \mathcal{X} complete, intrinsic, $\dim_{\text{Haus}} \mathcal{X} < \infty$.¹

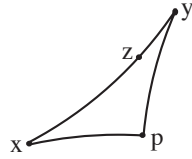
(1) (a) We say $\mathcal{X} \in \text{CAT}^\kappa \Leftrightarrow$

Point-Side

 holds:

if $\tilde{\Delta}^\kappa[pxy] = [\tilde{p}\tilde{x}\tilde{y}]$ is defined,
 $z \in [xy]$, $\tilde{z} \in [\tilde{x}\tilde{y}]$, $|xz| = |\tilde{x}\tilde{z}|$, then

$$(*) \quad |pz| \leq |\tilde{p}\tilde{z}|.$$



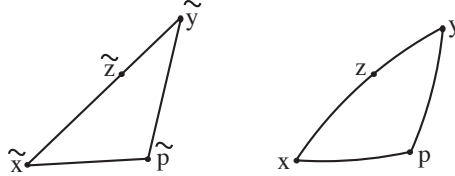
(1) (a) We say $\mathcal{X} \in \text{CBB}^\kappa \Leftrightarrow$

Point-Side

 holds:

if $\tilde{\Delta}^\kappa[pxy] = [\tilde{p}\tilde{x}\tilde{y}]$ is defined,
 $z \in [xy]$, $\tilde{z} \in [\tilde{x}\tilde{y}]$, $|xz| = |\tilde{x}\tilde{z}|$, then

$$(*) \quad |pz| \geq |\tilde{p}\tilde{z}|.$$



$$(b) \quad (*) \Leftrightarrow \tilde{Z}^\kappa[x_z^p] \leq \tilde{Z}^\kappa[x_y^p].$$

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(c) *Examples:* $\mathcal{X} =$

Tree; locally- \mathbf{E}^2 sector with any angle;
 locally- \mathbf{E}^2 cone with angle $\geq 2\pi$.

(c) *Examples:* $\mathcal{X} =$

Closed convex subset of \mathbf{E}^n ;
 locally- \mathbf{E}^2 cone with angle $\leq 2\pi$.

(d) Geodesics can bifurcate.

(d) Geodesics cannot bifurcate.

(e) Geodesics are unique.

(e) Geodesics need not be unique.

(2) (a) By monotonicity in (1) (b), for a hinge $[px] \cup [py]$:

Point-Side

 $\Rightarrow \angle[p_y^x]$ exists.

Point-Side

 $\Rightarrow \angle[p_y^x]$ exists.

(b) Also by (1) (b), Point-Side \Rightarrow the following Angle comparisons:

$$\angle[x_y^p] \leq \tilde{Z}^\kappa[x_y^p].$$

$$\angle[x_y^p] \geq \tilde{Z}^\kappa[x_y^p].$$

Exercise 2.1. *Semicontinuity of angles.* If $[p_n^{x_n}] \rightarrow [p^x]$ in \mathcal{X} , then

$\mathcal{X} \in \text{CAT}^\kappa \Rightarrow$

$$\angle[p_y^x] \geq \limsup_{n \rightarrow \infty} \angle[p_n^{x_n}].$$

$\mathcal{X} \in \text{CBB}^\kappa \Rightarrow$

$$\angle[p_y^x] \leq \liminf_{n \rightarrow \infty} \angle[p_n^{x_n}].$$

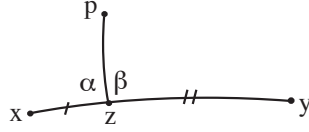
Prove and give examples.

Exercise 2.2. In $\mathcal{X} \in \text{CAT}^\kappa$, show a local geodesic of length $\leq \pi^\kappa$ is a geodesic.

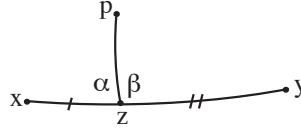
¹This assumption implies \mathcal{X} is proper, hence π^κ -geodesic. Infinite-dimensional theory is carried out in [AKP].

(3) Proof that Angle \Rightarrow Point-Side:

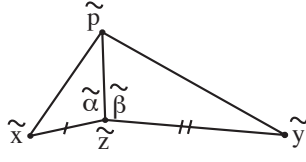
(a) $\mathcal{X} \in \text{CAT}^\kappa$:



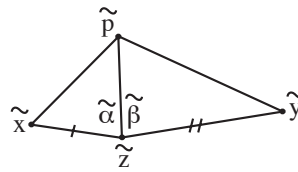
(a) $\mathcal{X} \in \text{CBB}^\kappa$:



(b) Side-by-side model triangles:



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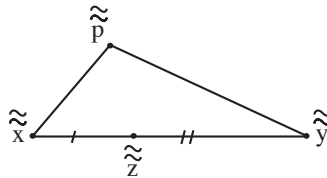
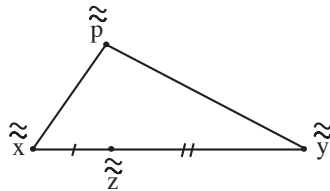


$\tilde{\alpha} + \tilde{\beta} \geq \alpha + \beta \geq \pi$ (by (2) (b) and triangle inequality for angles).

$\tilde{\alpha} + \tilde{\beta} \leq \alpha + \beta \leq \pi$ (by (2) (b) and ???).

(c) The triangle inequality for angles never switches sign! So in CBB^κ column of (2) (b), add to Angle: (**) $\alpha + \beta = \pi$. (You can check that Point-Side \Rightarrow (**).)

(d) Apply *Alexandrov Lemma* in $\mathcal{M}(\kappa)$ (think of 4 hinged rods in the plane):



“Snap out” to $\tilde{\Delta}^\kappa[pxy]$:
 $|pz| = |\tilde{p}\tilde{z}| \leq |\tilde{p}\tilde{\tilde{z}}|$.

“Snap in” to $\tilde{\Delta}^\kappa[pxy]$:
 $|pz| = |\tilde{p}\tilde{z}| \geq |\tilde{p}\tilde{\tilde{z}}|$. □

(e) *Remark*: For both “snap out” and “snap in”,

$$\tilde{Z}^\kappa[p_z^x] + \tilde{Z}^\kappa[p_y^z] \leq \tilde{Z}^\kappa[p_y^x].$$

Exercise 2.3. Referring to the figures in (3) (d), (b), (a):

(i) For $\mathcal{X} \in \text{CAT}^\kappa$, find a short, i.e. 1-Lipschitz, map

$$\text{ConvexHull} [\tilde{p}\tilde{x}\tilde{y}] \rightarrow \mathcal{M}(\kappa)$$

that maps $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z}$ to $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z}$.

(ii) We want a short map

$$\text{ConvexHull} [\tilde{p}\tilde{x}\tilde{y}] \rightarrow \mathcal{X}$$

that maps $\tilde{p}, \tilde{x}, \tilde{y}$ to p, x, y (hence maps $[\tilde{p}\tilde{x}\tilde{y}]$ to $[pxy]$). Check that the simplest map you might try is not short, i.e. the *line-of-sight* map taking geodesics that join \tilde{p} to points of $[\tilde{x}\tilde{y}]$ proportionally to geodesics that join p to corresponding points of $[xy]$.

(iii) Outline of how to get the short map in (ii) (not part of the exercise): Extend your construction in (i) to partition points $\tilde{z}^1, \dots, \tilde{z}^n$. In limit, get short map

$$\text{ConvexHull} [\tilde{p}\tilde{x}\tilde{y}] \rightarrow \mathcal{M}(\kappa)$$

mapping $\tilde{p}, \tilde{x}, \tilde{y}$ to $\tilde{p}, \tilde{x}, \tilde{y}$, and $\text{ConvexHull} [\tilde{p}\tilde{x}\tilde{y}]$ to “development of $[xy]$ from p ”. It is the line-of-sight map from this development to \mathcal{X} that is short.

(4) Exercise 2.3 is special case, where $\alpha = \text{triangle}$, of:

(a) *Reshetnyak majorization theorem* [R68]. Suppose $\mathcal{X} \in \text{CAT}^\kappa$, and α is a closed curve in \mathcal{X} of length $< 2 \cdot \pi^\kappa$. Then there is a convex set $\text{CS} \subset \mathcal{M}(\kappa)$ bounded by a curve $\tilde{\alpha}$, and a short map $F: \text{CS} \rightarrow \mathcal{X}$, such that $F \circ \tilde{\alpha}$ is length-preserving and $F \circ \tilde{\alpha} = \alpha$.

(5) Lang-Schroeder [LS97] developed the theory of extendibility of short maps

$$f : S \subset \mathcal{X}^1 \in \text{CBB}^\kappa \rightarrow \mathcal{X}^2 \in \text{CAT}^\kappa$$

(also see [AKP11]). e.g. characterizations of CBB^κ , CBB^κ :

(a) Suppose \mathcal{X} complete, intrinsic, $\dim_{\text{Haus}} \mathcal{X} < \infty$:

$\mathcal{X} \in \text{CBB}^\kappa \iff$ for any 3-point set V_3 and 4-point set V_4 , $V_3 \subset V_4 \subset \mathcal{X}$, any short map $f : V_3 \rightarrow \mathcal{M}(\kappa)$ extends to a short map $F : V_4 \rightarrow \mathcal{M}(\kappa)$.

(b) Suppose \mathcal{X} complete, and \exists unique $[xy]$ for any $x, y \in \mathcal{X}$ with $|xy| < \pi^\kappa$:

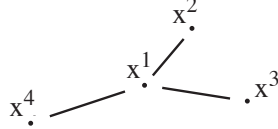
$\mathcal{X} \in \text{CAT}^\kappa \iff$ for any 3-point set V_3 and 4-point set V_4 , $V_3 \subset V_4 \subset \mathcal{M}(\kappa)$, where perimeter $V_3 < 2 \cdot \pi^\kappa$, any short map $f : V_3 \rightarrow \mathcal{X}$ extends to a short map $F : V_4 \rightarrow \mathcal{X}$.

(c) “ \Leftarrow ” : In both (a) and (b), extendibility condition immediately implies Point-Side.

(6) (a) A quadruple of points x^1, x^2, x^3, x^4 in a metric space \mathcal{X} satisfies $(1+3)$ -point κ -comparison, in brief $\boxed{(1+3)^\kappa}$, if

$$\tilde{Z}^\kappa[x^1 x^2_{x^3}] + \tilde{Z}^\kappa[x^1 x^3_{x^4}] + \tilde{Z}^\kappa[x^1 x^4_{x^2}] \leq 2 \cdot \pi,$$

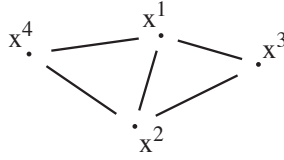
or at least one of the three model angles $\tilde{Z}^\kappa[x^1 x^i_{x^j}]$ is undefined [BGP92].



(b) A quadruple of points x^1, x^2, x^3, x^4 in a metric space \mathcal{X} satisfies $(2+2)$ -point κ -comparison, in brief $\boxed{(2+2)^\kappa}$, if

- (i) either $\tilde{Z}^\kappa[x^1 x^3_{x^4}] \leq \tilde{Z}^\kappa[x^1 x^3_{x^2}] + \tilde{Z}^\kappa[x^1 x^2_{x^4}]$,
- (ii) or $\tilde{Z}^\kappa[x^2 x^3_{x^4}] \leq \tilde{Z}^\kappa[x^2 x^3_{x^1}] + \tilde{Z}^\kappa[x^2 x^1_{x^4}]$,

or at least one of the six model angles $\tilde{Z}^\kappa[x^i x^j_{x^k}]$ is undefined [AKP].



(c) $\mathcal{X} \in \text{CAT}^\kappa \Leftrightarrow$
quadruples satisfy $\boxed{(2+2)^\kappa}$ [AKP].

(c) $\mathcal{X} \in \text{CBB}^\kappa \Leftrightarrow$
quadruples satisfy $\boxed{(1+3)^\kappa}$ [BGP92].

(d) These definitions $\Rightarrow \text{CAT}^\kappa, \text{CBB}^\kappa$ closed under GH-convergence to a complete metric space.

Exercise 2.4. Show that a complete π^κ -intrinsic space in which quadruples satisfy $\boxed{(2+2)^\kappa}$ is π^κ -geodesic.

3. GLOBALIZATION I

Globalization theorems are power sources for Alexandrov geometry. We look at extensions of these theorems and ideas of proof in a later section.

- (1) Let \mathcal{X} be a locally intrinsic space.
 - (a) \mathcal{X} has *curvature* $\geq \kappa$, written $\text{curv } \mathcal{X} \geq \kappa$, if all quadruples lying in some neighborhood of p satisfy $(1 + 3)$ -point κ -comparison;
 - (b) \mathcal{X} has *curvature* $\leq \kappa$, written $\text{curv } \mathcal{X} \leq \kappa$, if all quadruples lying in some neighborhood of p satisfy $(2 + 2)$ -point κ -comparison.

- (2) (a) Lemma: $\mathcal{X} \in \text{CBB}^\kappa$, $\dim \mathcal{X} = 1 \Leftrightarrow \mathcal{X}$ isometric to circle or closed, connected subset of \mathbf{R} .
- (b) *Burago-Gromov-Perelman CBB globalization theorem* [BGP92]². Let \mathcal{X} be a complete intrinsic space with $\text{curv } \mathcal{X} \geq \kappa$. Then $\mathcal{X} \in \text{CBB}^\kappa$. If $\dim \mathcal{X} > 1$, all triangles have perimeter $\leq 2 \cdot \pi^\kappa$.
- (c) This theorem was proved by Alexandrov in dimension 2 [A46], and by Toponogov for n -dimensional Riemannian manifolds [T59].
- (d) Only now (with triangles of perimeter $> 2 \cdot \pi^\kappa$ ruled out) can we prove: If $\mathcal{X} \in \text{CBB}^\kappa$ then $\mathcal{X} \in \text{CBB}^{\kappa'}$ for all $\kappa' \leq \kappa$.

- (3) (a) Upper curvature bounds yield more complicated globalization.
- (b) Ex. 2.3 (a), applied repeatedly to a patchwork of small triangles, gives: *Alexandrov patchwork*. $\mathcal{X} \in \text{CAT}^\kappa \Leftrightarrow \text{curv } \mathcal{X} \leq \kappa$ and geodesics joining points at distance $< \pi^\kappa$ exist, are unique, and vary continuously with their endpoint pairs.
- (c) *Hadamard-Cartan theorem*.³ If \mathcal{X} is a complete, simply connected intrinsic space satisfying $\text{curv } \mathcal{X} \leq \kappa \leq 0$, then $\mathcal{X} \in \text{CAT}^\kappa$.

- (4) (a) Complete intrinsic spaces of $\text{curv} \geq \kappa$ are closed under GH-convergence to a complete metric space, by CBB^κ globalization theorem and §2.(6)(d).
- (b) Not so for $\text{curv} \leq \kappa$! e.g. one-sheeted hyperboloids with waist circumferences $\rightarrow 0$ satisfy $\text{curv} \leq 0$. GH-limit = double cone (which does not have unique geodesics in any neighborhood of the vertex).

²This theorem does not assume finite dimension.

³The Hadamard-Cartan theorem was first stated for geodesic spaces \mathcal{X} by Gromov [G87, p.119]. A detailed proof when \mathcal{X} is proper was given by Ballmann [Ba85]. A proof in the non-proper case was given by Alexander-Bishop [AB90]. Kleiner, Bridson-Haefliger observed that this proof extends to intrinsic spaces.

4. CBB APPLICATION: ORBIT SPACE

Exercise 4.1. [BGP92] Suppose $\text{curv } \tilde{\mathcal{X}} \geq \kappa$ and a group Γ acts on $\tilde{\mathcal{X}}$ by isometries with closed orbits. Show $\text{curv}(\tilde{\mathcal{X}}/\Gamma) \geq \kappa$.

(1) (a) *Danzer-Grünbaum theorem* [DG62]. (*Erdős' problem*[E57].) If $x^1, \dots, x^m \in \mathbf{E}^n$ have all angles $\leq \pi/2$, then $m \leq 2^n$. If $m = 2^n$, the x^i are vertices of a right parallelepiped.

(b) *Danzer-Grünbaum-Perelman theorem*. If $\mathcal{X} \in \text{CBB}^0$, $\dim \mathcal{X} = n$, $m = \#$ extremal points, then $m \leq 2^n$.

(c) *Corollary*. For a discrete isometric action on \mathbf{E}^n , the number of isolated singular orbits is $\leq 2^n$.

(2) (a) Here p is an *extremal point* of $\mathcal{X} \in \text{CBB}^\kappa$ if the “unit tangent sphere” $\Sigma_p \mathcal{X}$ has diameter $\leq \pi/2$ (see §8).

(b) Given a discrete isometric action by Γ on \mathbf{E}^n , $\mathcal{X} = \mathbf{E}^n/\Gamma \in \text{CBB}^0$. An extremal point of \mathcal{X} corresponds to an isolated singular orbit, i.e. the image of an isolated fixed point in \mathbf{E}^n of some subgroup of Γ .

(c) *Proof of Corollary*. Immediate from (b) and D-G-P theorem.

(d) *Lebedeva's theorem* [Lb11]. If $\mathcal{X} \in \text{CBB}^0$ has dimension n and 2^n extremal points, \mathcal{X} is isometric to \mathbf{E}^n/Γ for a discrete, cocompact isometric action by Γ .

(e) Examples of (d): Euclidean rectangular solid; tetrahedral surface glued from four congruent Euclidean triangles.⁴

⁴Source for this section: [Lb11].

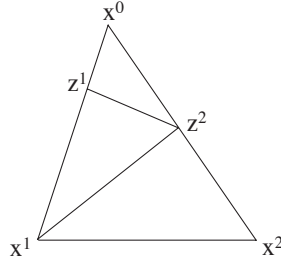
5. CAT CONSTRUCTION: GLUING

(1) (a) We say $CS \subset \mathcal{X}$ is π^κ -convex if $|xy| < \pi^\kappa \Rightarrow \exists [xy]$, and all $[xy] \subset CS$.

(b) CAT^κ is closed under gluing on isometric π^κ -convex subsets:

Reshetnyak gluing theorem [R60]. Consider $\mathcal{X}^1, \mathcal{X}^2 \in CAT^\kappa$, closed π^κ -convex subsets $CS^i \subset \mathcal{X}^i$, and an isometry $\iota : CS^1 \rightarrow CS^2$ of CS^1 onto CS^2 . If $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$ is the intrinsic space obtained by gluing \mathcal{X}^i along ι , then $\mathcal{X} \in CAT^\kappa$.

(c) *Proof assuming \mathcal{X} is geodesic.* For triangle $[x^0x^1x^2]$ of perimeter $< 2 \cdot \pi^\kappa$ in \mathcal{X} , we may suppose $x^0 \in \mathcal{X}^1$ and $x^1, x^2 \in \mathcal{X}^2$. Choose points z^1, z^2 in the gluing set, and lying on sides $[x^0x^1]$ and $[x^0x^2]$ respectively. Then $[x^0z^1z^2] \subset \mathcal{X}^1$, and $[x^1z^1z^2] \subset \mathcal{X}^2$, $[x^1z^2x^2] \subset \mathcal{X}^2$. Now apply Alexandrov “snap-out” Lemma twice to side-by-side model triangles.



Exercise 5.1. (Bishop [Bi08], Petrunin)

- (i) Let \mathcal{P} be a polygon in \mathbf{E}^2 carrying its intrinsic metric. Show $\mathcal{P} \in CAT^0$.
- (ii) Let \mathcal{X} be the completion in the intrinsic metric of an open, simply connected, proper subset $\check{\mathcal{X}}$ of \mathbf{E}^2 . Show $\mathcal{X} \in CAT^0$.

(2) (a) Dually, CBB^κ is closed under gluing on isometric boundaries, e.g. closed balls in \mathbf{E}^n and \mathbf{H}^n , of radius $\sinh r$ and r respectively, glued on their (isometric) boundary spheres.

Petrunin gluing theorem [P97]. Consider $\mathcal{X}^1, \mathcal{X}^2 \in CBB^\kappa$, and an isometry $\iota : \partial\mathcal{X}^1 \rightarrow \partial\mathcal{X}^2$ of $\partial\mathcal{X}^1$ onto $\partial\mathcal{X}^2$. If $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$ is the intrinsic space obtained by gluing \mathcal{X}^i along ι , then $\mathcal{X} \in CBB^\kappa$.

- (b) If $\mathcal{X}^1 = \mathcal{X}^2$, this is *Perelman doubling theorem* [Pr91].
- (c) CBB^κ gluing proof is harder than CAT^κ !
- (d) Definition of $\partial\mathcal{X}$ for $\mathcal{X} \in CBB^\kappa$ is given in §11.(5).

6. CAT APPLICATION: BILLIARDS

Burago, Ferleger, and Kononenko [BFK98] used CAT^0 geometry to solve a long-standing, celebrated open problem: *In a gas of m hard balls, \exists a uniform bound on number of collisions.* Their method is by solving a more general billiards problem.⁵

(1) (a) Configuration space for m balls of radius $1/2$ in \mathbf{E}^3 :

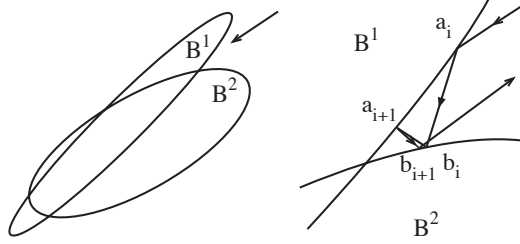
$$\bar{\mathcal{S}} = \mathbf{E}^{3m} - \{(x^1, \dots, x^m) \in \mathbf{E}^{3m} : |x^i - x^j| < 1\}.$$

(b) Collisions $\sim \partial\bar{\mathcal{S}}$, motions of the balls \sim billiards trajectories on table $\bar{\mathcal{S}}$.

(2) (a) More generally, for *any* convex bodies $B^i \subset \mathbf{E}^N$, consider billiards in

$$\bar{\mathcal{S}} = \mathbf{E}^N - \cup_{i=1}^m B^i.$$

For simplicity, let $m = 2$, $\bar{\gamma}$ = billiard trajectory in $\bar{\mathcal{S}} = \mathbf{E}^N - (B^1 \cup B^2)$.



(b) Each time $\bar{\gamma}$ strikes ∂B^i , say at p , glue a new copy of $\mathbf{E}^N - B^i$ along ∂B^i . Let \mathcal{S} be the resulting space. Then $\bar{\gamma}$ is the projection of a local geodesic γ in \mathcal{S} that moves from one leaf to the next at each p .

(c) \mathcal{S} is now our billiard table. Alas, \mathcal{S} takes on curvature of ∂B^i ! But by Reshetnyak gluing (§5), gluing in one copy of each B^i gives

$$\mathcal{S} \subset \mathcal{X} \in \text{CAT}^0!$$

(d) γ remains a local geodesic in the new space \mathcal{X} , hence a geodesic by Ex. 2.2. Goal: upper bound on $\#$ (strike points of γ). (Difficult case: γ almost parallel to intersection curve of B^1, B^2 .)

(3) Core of proof (local argument): Assume γ lies in a neighborhood in \mathcal{S} satisfying a necessary nondegeneracy condition on tangent planes to ∂B_i (see (b)).

(a) Let $a_1, \dots, a_n \in B^1$ and $b_1, \dots, b_n \in B^2$ be the strike points of γ (see figure). Set $d_i = |a_i b_i| + |b_i a_{i+1}| - |a_i a_{i+1}|$, and suppose d_j is the least of these.

(b) Let $z_i \in B^1 \cap B^2$ be the closest point of $B^1 \cap B^2$ to a_i . By nondegeneracy condition, $\exists C$ such that replacing $[a_i b_i] \cup [b_i a_{i+1}]$ by $[a_i z_i] \cup [z_i a_{i+1}]$ lengthens γ by $\leq C \cdot d_i$.

(c) Define a new path by replacing:

- $[a_j b_j] \cup [b_j a_{j+1}]$ by $[a_j z_j] \cup [z_j a_{j+1}]$;
- $[a_i b_i] \cup [b_i a_{i+1}]$ by $[a_i, a_{i+1}]$ for all $i \neq j$.

(d) Since new path intersects $B^1 \cap B^2$, length strictly increases. Hence $C \cdot d_j - (n-1) \cdot d_j > 0$, so $n < C + 1$.

⁵Source for this section: [BBI, §9.4]. The theorem applies to non-negatively curved Riemannian manifolds, not just to \mathbf{E}^n .

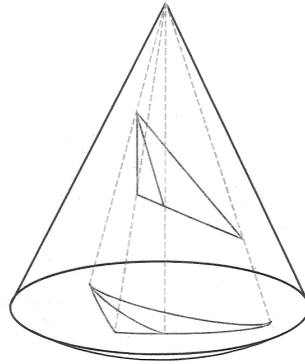
7. CONES

- Let \mathcal{X} be a metric space.

Definition 7.1. Cone $\mathcal{X} = (\mathcal{X} \times [0, \infty)) / (\mathcal{X} \times \{0\})$ with metric

$$|(x, s)(y, t)| = \sqrt{s^2 + t^2 - 2 \cdot s \cdot t \cdot \cos(\min\{|x - y|, \pi\})}.$$

Exercise 7.2. Show $\mathcal{X} \in \text{CAT}^1 \iff \text{Cone } \mathcal{X} \in \text{CAT}^0$.



Similarly: If $\mathcal{X} \in \text{CBB}^1$ and $\mathcal{X} \neq$ connected subset of \mathbf{R} of length $> \pi^\kappa$ or circle of length $> 2 \cdot \pi^\kappa$, then $\text{Cone } \mathcal{X} \in \text{CBB}^0$. If $\text{Cone } \mathcal{X} \in \text{CBB}^0$, then $\mathcal{X} = 2$ points or \mathcal{X} is connected and $\mathcal{X} \in \text{CBB}^1$.

8. FIRST ORDER STRUCTURE

- Suppose $\mathcal{X} \in \text{CAT}^\kappa$ or $\mathcal{X} \in \text{CBB}^\kappa$.

(1) (a) Set $\Gamma_p\mathcal{X} =$ geodesics γ with $\gamma(0) = p$. Set $\gamma_1 \sim \gamma_2$ if this angle is 0. The *space of geodesic directions* is

$$\Sigma'_p\mathcal{X} = (\Gamma_p\mathcal{X} / \sim)$$

carrying the angle metric.

(b) The *direction space* $\Sigma_p\mathcal{X}$ is the metric completion of $\Sigma'_p\mathcal{X}$.

(c) The *tangent space* of \mathcal{X} at p is $T_p\mathcal{X} = \text{Cone}(\Sigma_p\mathcal{X})$.

(2) If $\text{curv } \mathcal{X} \leq \kappa$, or $\text{curv } \mathcal{X} \geq \kappa$, \mathcal{X} locally compact and all geodesics of \mathcal{X} extendible, then

$$T_p\mathcal{X} = \text{GH-lim}_{\lambda \rightarrow \infty} (\lambda\mathcal{X}, p).$$

Exercise 8.1. Give examples of $\mathcal{X} \in \text{CAT}^\kappa$ and $p \in \mathcal{X}$, where:

- (i) the limit in (2) does not exist;
- (ii) the limit in (2) exists but is not a cone over a metric space.

Theorem.

- (i) [BGP92] $\text{curv } \mathcal{X} \geq \kappa \Rightarrow T_p\mathcal{X} \in \text{CBB}^0$.
- (ii) [N95] $\text{curv } \mathcal{X} \leq \kappa \Rightarrow T_p\mathcal{X} \in \text{CAT}^0$.

9. RIEMANNIAN EXAMPLES

- Let \mathcal{X} = n -dimensional Riemannian manifold ($n \geq 2$), possibly with $\partial\mathcal{X} \neq \emptyset$.
- For $p \in \partial\mathcal{X}$, denote the principal curvatures of $\partial\mathcal{X}$ at p by

$$k_1(p) \leq k_2(p) \leq \dots \leq k_{n-1}(p).$$

Convention: $k_i(p) > 0$ for \mathcal{X} = closed ball in \mathbf{E}^n ;
 $k_i(p) < 0$ for \mathcal{X} = complement of open ball in \mathbf{E}^n .

(1) (a) $\text{curv } \mathcal{X} \geq \kappa \Leftrightarrow \text{sec } \mathcal{X} \geq \kappa$, and $\partial\mathcal{X}$ convex (all $k_i(p) \geq 0$).
(Sec denotes sectional curvature.)

(b) *Buyalo convex hypersurface theorem* [B76, AKP08]. Suppose $\partial\mathcal{X} = \emptyset$, $\text{sec } \mathcal{X} \geq \kappa$, and $\mathcal{C} \subset \mathcal{X}$ is a convex hypersurface with its intrinsic metric, i.e. \mathcal{C} bounds a convex set. Then $\mathcal{C} \in \text{CBB}^\kappa$.

(c) *Kosovskii gluing theorem for curv $\geq \kappa$* [Ko02-1]. Consider $\mathcal{X}^1, \mathcal{X}^2$ with $\text{sec } \mathcal{X}^i \geq \kappa$, and an isometry $\iota: \partial\mathcal{X}^1 \rightarrow \partial\mathcal{X}^2$ of $\partial\mathcal{X}^1$ onto $\partial\mathcal{X}^2$. Set $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$. Then $\text{curv } \mathcal{X} \geq \kappa \Leftrightarrow II(\partial\mathcal{X}^1) + II(\partial\mathcal{X}^2)$ is positive semi-definite.

(2) (a) *A.-Berg-Bishop characterization theorem for manifolds-with-boundary* [ABB93]. $\text{curv } \mathcal{X} \leq \kappa \Leftrightarrow \text{sec } \mathcal{X} \leq \kappa$ and $\text{sec}(\partial\mathcal{X})(P) \leq \kappa$ for any tangent 2-plane P on which $\partial\mathcal{X}$ is strictly concave, i.e. the second fundamental form $II(\partial\mathcal{X})$ is negative definite.

(b) *Corollary*. Suppose $\mathcal{X} \subset \bar{\mathcal{X}}$ where $\bar{\mathcal{X}}$ is an n -dimensional Riemannian manifold with $\text{sec } \bar{\mathcal{X}} \leq \kappa$. If $k_2(p) \geq 0$ for all $p \in \partial\mathcal{X}$, then $\text{curv } \mathcal{X} \leq \kappa$.

(c) *Kosovskii gluing theorem for curv $\leq \kappa$* [Ko02-2, Ko04]. Consider $\mathcal{X}^1, \mathcal{X}^2$ as in (2)(a), with $\text{curv } \mathcal{X}^i \leq \kappa$, and an isometry $\iota: \partial\mathcal{X}^1 \rightarrow \partial\mathcal{X}^2$ of $\partial\mathcal{X}^1$ onto $\partial\mathcal{X}^2$. Set $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$. If $II(\partial\mathcal{X}^1) + II(\partial\mathcal{X}^2)$ is negative semi-definite, then $\text{curv } \mathcal{X} \leq \kappa$. The theorem extends to multiple \mathcal{X}^i all of whose whose boundaries are isometric.

10. CONVEX FUNCTIONS

• Let \mathcal{X} be a metric space.

(1) (a) For locally Lipschitz $f : \mathcal{X} \rightarrow \mathbf{R}$, write

$$f'' + \kappa f \geq c \quad (\leq c)$$

if (*) $(f \circ \gamma)'' + \kappa \cdot (f \circ \gamma) \geq c$ ($\leq c$) for every unitspeed geodesic γ .

(b) (*) is in barrier sense, i.e. for each t , some y such that $y'' + \kappa \cdot y = c$ coincides with $(f \circ \gamma)$ at t , and satisfies $(f \circ \gamma) \geq y$ (respectively $\leq y$) on a sufficiently short open interval about t (*support property*).

(c) Then f is *semiconvex* (*semiconcave*): \exists constant generalized lower (upper) bound on f'' along geodesics. (f semiconvex \Rightarrow $(-f)$ semiconcave.)

(2) (a) Set

$$\text{md}^\kappa(x) = \begin{cases} x^2/2, & \kappa = 0 \\ (1/\kappa)(1 - \cos \sqrt{\kappa}x), & \kappa > 0 \\ (1/\kappa)(1 - \cosh \sqrt{-\kappa}x), & \kappa < 0. \end{cases}$$

$$\text{sn}^\kappa(x) = (\text{md}^\kappa)'(x) = \begin{cases} x, & \kappa = 0 \\ (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}x), & \kappa > 0 \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}x), & \kappa < 0. \end{cases}$$

(b) In $\mathcal{M}^2(\kappa)$, $f = \text{md}^\kappa \circ \text{dist}_p$ satisfies $f'' + \kappa f = 1$.

(c) In $\mathcal{M}(\kappa)$, $f = \text{sn}^\kappa \circ \text{dist}_{[xy]}$ satisfies $f'' + \kappa f \geq 0$.

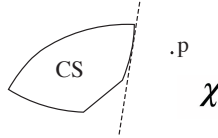
(3) Set $B(A, r) = \{q \in \mathcal{X} : |qA| < r\}$ for $A \subset \mathcal{X}$. By (2)(b) and Point-Side,

(a) $\mathcal{X} \in \text{CAT}^\kappa \Leftrightarrow f = \text{md}^\kappa \circ \text{dist}_p$ satisfies $f'' + \kappa f \geq 1$ on $B(p, \pi^\kappa)$.

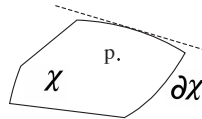
(b) $\mathcal{X} \in \text{CBB}^\kappa \Leftrightarrow f = \text{md}^\kappa \circ \text{dist}_p$ satisfies $f'' + \kappa f \leq 1$ on $B(p, \pi^\kappa)$.

(4) (a) On the other hand, (2)(c) relates to the following theorems:

(b) *Theorem.* Let $\mathcal{X} \in \text{CAT}^\kappa$, and $\text{CS} \subset \mathcal{X}$ be π^κ -convex. Then $f = \text{sn}_\kappa \circ \text{dist}_{\text{CS}}$ satisfies $f'' + \kappa f \geq 0$ on $B(\text{CS}, \pi^\kappa/2)$.



(c) *Perelman concavity theorem.* If $\mathcal{X} \in \text{CBB}^\kappa$, then $f = \text{sn}_\kappa \circ \text{dist}_{\partial\mathcal{X}}$ satisfies $f'' + \kappa f \leq 0$. (See §11.(5) for definition of $\partial\mathcal{X}$.)



(5) (a) *Lemma.* Let $\text{curv } \mathcal{X} \geq \kappa$ or $\text{curv } \mathcal{X} \leq \kappa$, and $f : X \rightarrow \mathbf{R}$ be a locally Lipschitz function such that $(f \circ \gamma)^+(0)$ exists for every geodesic γ with $\gamma(0) = p$. Then $\exists!$ linearly homogeneous, Lipschitz map

$$d_p f : T_p X \rightarrow \mathbf{R}$$

such that $(d_p f)(x) = (f \circ \gamma)^+(0)$ when γ is a geodesic with $\gamma^+(0) = x$.

(b) Let $\text{curv } \mathcal{X} \geq \kappa$ or $\text{curv } \mathcal{X} \leq \kappa$, and $f : X \rightarrow \mathbf{R}_{\geq 0}$ be a locally Lipschitz semiconcave function ((1)(c)). Then:

(i) $d_p f$ exists and is concave.

(ii) The *gradient* $\nabla_p f \in T_p X$ exists, where $\nabla_p f = 0_p$ if $d_p f \leq 0$, and otherwise

$$\nabla_p f = (d_p f)(u_{\max}) \cdot u_{\max},$$

for unique $u_{\max} \in \Sigma_p \mathcal{X}$ where $(d_p f) | \Sigma_p \mathcal{X}$ takes its maximum.

(c) Semiconcave functions capture much of the geometry of Alexandrov spaces. Their *gradient curves* exist and are unique, and the corresponding gradient flow is locally Lipschitz. Gradient curves of semiconcave functions were introduced in [PP94] (for $\text{curv} \geq \kappa$), and their properties developed by Lytchak [Lt 05] (for both $\text{curv} \geq \kappa$ and $\text{curv} \leq \kappa$) and Petrunin [P07].

Exercise 10.1. Let \mathcal{X} be a closed triangular region in \mathbf{E}^2 . Describe the gradient curves of $f = \text{dist}_{\partial X}$.

11. LOCAL STRUCTURE

- (1) (a) Notions of metric space *dimension* include Hausdorff, topological (covering), and *geometric*: the largest number of times you can pass to direction space without getting the empty set.
- (2) $\mathcal{X} \in \text{CBB}^\kappa$ has nice local structure [BGP92, Pr91, Pr93] :
- (a) \mathcal{X} is locally compact, hence proper and geodesic.
 - (b) $\dim_{\text{Haus}} \mathcal{X}$ is an integer.
 - (c) All extant notions of $\dim \mathcal{X}$ agree.
 - (d) There is an open dense set of points p such that p has a neighborhood bi-Lipschitz homeomorphic to an open region in \mathbf{R}^n .
 - (e) If $\dim \mathcal{X} = n$, then $T_p \mathcal{X} \in \text{CBB}^0$ has dimension n and $\Sigma_p \mathcal{X} \in \text{CBB}^1$ has dimension $n - 1$ for all $p \in \mathcal{X}$.
 - (f) The set of points p at which $\Sigma_p \mathcal{X}$ is isometric to \mathbf{S}^{n-1} is dense, and a countable intersection of open dense sets.
- (3) Kleiner [K199] introduced geometric dimension and studied its relation to other notions of $\dim \mathcal{X}$ for $\mathcal{X} \in \text{CAT}^\kappa$ (a more complicated situation than CBB^κ). e.g. Conjecture: $\dim_{\text{Geom}} \mathcal{X} = \dim_{\text{Top}} \mathcal{X}$ (true for separable \mathcal{X}).

Exercise 11.1. Construct a metric tree \mathcal{X} such that \mathcal{X}^c has infinite Hausdorff dimension, where \mathcal{X}^c is the metric completion of \mathcal{X} . (Note that $\mathcal{X}^c \in \text{CAT}^0$ by Ex. 2.4. Also $\dim_{\text{Top}} \mathcal{X} = 1$.)

- (4) (a) A key tool in (3) is the *barycentric simplex* Θ_f of $p^0, \dots, p^k \in \mathcal{X}$ [K199]. Set $f = (f^0, \dots, f^k)$, $f^i = \text{md}^\kappa \circ \text{dist}_{p^i}$ (§10.(3)(a)). Let $\Delta^k = \{\mathbf{x} \in \mathbf{R}^k : \sum_{i=0}^k x^i = 1, x^i \geq 0\}$, $\mathbf{x} = (x^0, \dots, x^k)$. Define $\Theta_f : \Delta^k \rightarrow \mathcal{X}$ by

$$\Theta_f(\mathbf{x}) = \text{MinPoint} \sum_{i=0}^k x^i \cdot f^i.$$

- (b) The theorems of (2) depend on analogous constructions, with detailed study of strutting point arrays (“strainers”) [BGP92, Pr91, Pr93].

- (5) Definition of $\partial \mathcal{X}$ for $\mathcal{X} \in \text{CBB}^\kappa$ (by induction on dimension):
- (a) If $\dim \mathcal{X} = 1$, then $\partial \mathcal{X} =$ topological boundary of \mathcal{X} (so $\partial \mathcal{X} = 1$ or 2 points, or $\partial \mathcal{X} = \emptyset$).
 - (b) If $\dim \mathcal{X} = n > 1$, then

$$\partial \mathcal{X} = \{p \in \mathcal{X} : \partial(\Sigma_p \mathcal{X}) \neq \emptyset\}.$$

12. WARPED PRODUCTS

(1) (a) Let \mathcal{B}, \mathcal{F} be intrinsic spaces, and $f: \mathcal{B} \rightarrow \mathbf{R}_{\geq 0}$ be continuous. Define $(\mathcal{B} \times \mathcal{F})/\sim$ by identifying elements of $\{p\} \times \mathcal{F}$ when $f(p) = 0$ (we identify this class with p). A curve $\gamma: J \rightarrow (\mathcal{B} \times \mathcal{F})/\sim$ determines $\gamma_{\mathcal{B}}: J \rightarrow \mathcal{B}$ and $\gamma_{\mathcal{F}}: J_+ \rightarrow \mathcal{F}$ for $J_+ = J - J_0, J_0 = (f \circ \gamma_{\mathcal{B}})^{-1}(0)$. Suppose $\gamma_{\mathcal{B}}$ and $\gamma_{\mathcal{F}}$ are Lipschitz.

(b) Writing $J_+ = \sqcup J_i$, set

$$\text{length } \gamma = \sum_i \int_{J_i} \sqrt{v_{\mathcal{B}}^2 + (f \circ \gamma_{\mathcal{B}})^2 \cdot v_{\mathcal{F}}^2} + \text{length}(\gamma_{\mathcal{B}}|_{J_0}),$$

where \int is Lebesgue integral, and $v_{\mathcal{B}}, v_{\mathcal{F}}$ are speeds of $\gamma_{\mathcal{B}}, \gamma_{\mathcal{F}}$ (defined almost everywhere on their domains).

(c) The *warped product* $\mathcal{X} = \mathcal{B} \times_f \mathcal{F}$ is the corresponding intrinsic space.

(2) *Examples:* (a) $\mathcal{B} \times_1 \mathcal{F} = \mathcal{B} \times \mathcal{F}$.

(b) $\mathcal{X} = \{r \in [0, c < \pi]\} \times_{\sin r} \mathbf{S}^n(1)$ is a metric ball in $\mathbf{S}^{n+1}(1)$.

$c \leq \pi/2 \Rightarrow \mathcal{X} \in \text{CBB}^1; \mathcal{X} \in \text{CAT}^1$.

$c > \pi/2 \Rightarrow \mathcal{X} \notin \text{CBB}^\kappa$ for any $\kappa; \mathcal{X} \in \text{CAT}^{1/\sin^2 c}$.

(c) κ -*cone of* \mathcal{F} : $\text{Cone}_\kappa \mathcal{F} = \{r \in \mathbf{I}^\kappa\} \times_{\text{sn}^\kappa r} \mathcal{F}, \mathbf{I}^\kappa = [0, \infty)$ if $\kappa \leq 0$, or $[0, \pi^\kappa]$.

Spherical suspension of \mathcal{F} : $\text{Cone}_1 \mathcal{F}$.

κ -*join of* $\mathcal{F}^1, \mathcal{F}^2$: $\text{Join}_\kappa(\mathcal{F}^1, \mathcal{F}^2) = (\{r \in \mathbf{I}^\kappa\} \times_{\text{sn}^\kappa r} \mathcal{F}^1) \times_{\text{cs}^\kappa \circ \text{proj}_r} \mathcal{F}^2$,
where $\text{cs}^\kappa = (\text{sn}^\kappa)'$, and proj_r is projection to first factor.

(d) Let $\mathcal{M}^0(\kappa)$ be a point if $\kappa \leq 0$; two points at distance π^κ if $\kappa > 0$. If $n \geq 1$,

$$\mathcal{M}^n(\kappa) = \text{Cone}_\kappa \mathcal{M}^{n-1}(1), \quad \mathcal{M}^{m+n+1}(\kappa) = \text{Join}_\kappa(\mathcal{M}^m(1), \mathcal{M}^n(\text{sgn } \kappa)).$$

Exercise 12.1. (i) Let \mathcal{B} be a closed triangular region in \mathbf{E}^2 . Describe $\mathcal{X} = \mathcal{B} \times_f \mathbf{S}^1(1)$, where $f = \text{dist}_{\partial \mathcal{B}}$.

(ii) Let $\mathcal{X} = \mathbf{E}^2 \times_f \mathbf{S}^1(1)$, where $f = \text{dist}_0$. Find $\Sigma_{\bar{0}} \mathcal{X}$.

Exercise 12.2. Show the spherical join, $\text{Join}_1(\mathcal{X}^1, \mathcal{X}^2)$, satisfies

$$\text{Cone } \text{Join}_1(\mathcal{X}^1, \mathcal{X}^2) = \text{Cone } \mathcal{X}^1 \times \text{Cone } \mathcal{X}^2.$$

(3) (a) $\mathcal{B} \times \{\varphi_0\}$ is isometric to \mathcal{B} , and isometrically embedded in \mathcal{X} .

(b) If $f(p_0) \neq 0$, then $\{p_0\} \times \mathcal{F}$, with its intrinsic metric, is homothetic to \mathcal{F} with multiplier $f(p_0)$.

(c) By length formula, geodesics of \mathcal{X} seek lower f -values. e.g. If $f(p_0) > 0$ is local min of f , then $\{p_0\} \times \mathcal{F}$ is isometrically embedded in \mathcal{X} . In this case, $\text{curv } \mathcal{X} \leq \kappa \Rightarrow \text{curv } \mathcal{F} \leq \kappa_{\mathcal{F}} = \kappa \cdot f(p_0)^2$, and same for \geq .

13. WARPED PRODUCT EXAMPLES

• Suppose $\mathcal{X} = \mathcal{B} \times_f \mathcal{F}$ for \mathcal{B}, \mathcal{F} intrinsic; $f : \mathcal{B} \rightarrow \mathbf{R}_{\geq 0}$ Lipschitz; $Z = f^{-1}(0) \neq \mathcal{B}$. Write $f \in \widehat{\mathcal{C}}^\kappa$ if $f'' + \kappa f \geq 0$, and $f \in \check{\mathcal{C}}^\kappa$ if $f'' + \kappa f \leq 0$.

(1) *A.-Bishop characterization theorem for warped products* [AB90, AB 98, AB13]:

$$\boxed{\mathcal{X} \in \text{CBB}^\kappa \iff :^6}$$

$$\boxed{\mathcal{X} \in \text{CAT}^\kappa \iff :}$$

(a) $\mathcal{B} \in \text{CBB}^\kappa$ and $f \in \widehat{\mathcal{C}}^\kappa$.

(a) $\mathcal{B} \in \text{CAT}^\kappa$ and $f \in \check{\mathcal{C}}^\kappa$.

(b) $\mathcal{F} \in \text{CBB}^{\kappa_{\mathcal{F}}}$ where:

(b) $\mathcal{F} \in \text{CAT}^{\kappa_{\mathcal{F}}}$ where:

(i) if $Z = \emptyset$, $\kappa_{\mathcal{F}} = \kappa \cdot (\inf f)^2$

(i) if $Z = \emptyset$, $\kappa_{\mathcal{F}} = \kappa \cdot (\inf f)^2$

(This condition is §12.(3)(c), or its asymptotic version.)

(ii) if $Z \neq \emptyset$, $\kappa_{\mathcal{F}} = \sup \{(f \circ \alpha)^+(0)^2\}$
for dist_Z -realizers α with
footpoint $\alpha(0) \in Z$, $|\alpha^+(0)| = 1$.

(ii) if $Z \neq \emptyset$, $\kappa_{\mathcal{F}} = \inf \{(f \circ \alpha)^+(0)^2\}$
for dist_Z -realizers α with
footpoint $\alpha(0) \in Z$, $|\alpha^+(0)| = 1$.

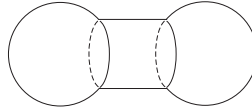
e.g. $\mathcal{X} = \text{Cone}_\kappa \mathbf{S}^n(1/c^2)$, $c \leq 1$.

e.g. $\mathcal{X} = \text{Cone}_\kappa \mathbf{S}^n(1/c^2)$, $c \geq 1$.

OR (if less)

(iii) $\kappa_{\mathcal{F}} = \inf \{\kappa \cdot f(p)^2 : \text{dist}_Z(p) \geq \varpi^\kappa/2\}$

(e.g. $\mathcal{X} = \text{dumbbell} = I \times_f \mathbf{S}^n(1)$, $n \geq 2$, $I = [0, c > \pi]$;
 $f(x) = \sin x$ for $x \leq \pi/2 + \epsilon < c/2$;
 $f(x) = \sin(x - (c - \pi))$ for $x \geq c - \pi/2 - \epsilon$; otherwise
 $f(x) = \sin(\pi/2 + \epsilon)$. For $\kappa = 1$, \mathcal{X} satisfies (1), (2)(ii),
not (2)(iii). $\mathcal{X} \notin \text{CAT}^1$ since $\{c/2\} \times \mathbf{S}^n(1)$ is
isometrically embedded in \mathcal{X} with curvature $1/f(c/2)^2 > 1$.)



(c) *Gluing condition:*

$\mathcal{B}^\dagger \in \text{CBB}^\kappa$, $f^\dagger \in \widehat{\mathcal{C}}^\kappa$ where $\mathcal{B}^\dagger =$
2 copies of \mathcal{B} glued on $\text{cl}(\partial\mathcal{B} - Z)$,
 $f^\dagger =$ canonical extension of f to \mathcal{B}^\dagger .

(e.g. $\tilde{\mathcal{B}} =$ closed unit ball in \mathbf{E}^n , $\tilde{f} = \text{dist}_{\partial\tilde{\mathcal{B}}}$, $\mathcal{B} =$ convex body in $\tilde{\mathcal{B}}$,
 $f = \tilde{f}|_{\mathcal{B}}$. Then \mathcal{B}, f satisfy gluing condition for $\kappa = 0 \iff$
 $\mathcal{B} \cap (\text{any ray from } 0) \text{ either has endpoint on } \partial\tilde{\mathcal{B}}, \text{ or } = \emptyset$.)

⁶For this theorem, exclude connected subsets of \mathbf{R} of length $> \pi^\kappa$ and circles of length $> 2 \cdot \pi^\kappa$ from CBB^κ .

(2) *Application of warped product characterization theorem:*

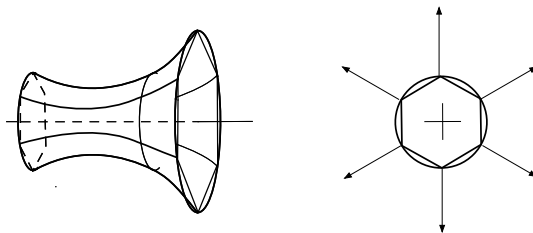
- (a) Gromov Q [G87]: If $\mathcal{X} \in \text{CAT}^0$ is an n -manifold, is \mathcal{X} homeomorphic to \mathbf{E}^n ?
- (b) Davis-Januszkiewicz gave counterexamples by (complicated) hyperbolization process [DJ92]. Ancel-Guilbault used hyperbolic trigonometry and structure theory for compact contractible manifolds to construct counterexamples [AG97]. In fact these are hyperbolic joins (Join_{-1}).
- (c) Q [AG97]: If $\mathcal{B} \in \text{CAT}^{-1}$, does $\mathcal{B} \times \mathbf{R}$ always carry a negatively curved metric whose levels are convex subsets?
- (d) *Yes:* $\mathcal{B} \times_f \mathbf{R} \in \text{CAT}^{-1}$ for any $f \in \check{\mathcal{C}}^{-1}$ (e.g. $f = \cosh d_p$ for $p \in \mathcal{B}$, or $f = \exp$ (Busemann function)). These are the only warped product examples.

(3) *Proving “if” in CAT^0 characterization theorem:*

- (a) Reduce to $\mathcal{F} = \mathcal{M}^2(\kappa)$. By §12.(2)(d), reduce to $\dim \mathcal{F} = 1$.
- (b) (Take $\kappa = 0$.) Decompose $B \times \mathbf{R}$ into *strips* carrying the metric of

$$\{(p, u) : -\epsilon f(p) \leq u \leq \epsilon f(p)\},$$

gluing top boundary of one strip to bottom boundary of next to get approximating space \mathcal{S} . If $\mathcal{B} \in \text{CBB}^0$ and f concave, then $\mathcal{S} \in \text{CBB}^0$. But when $\mathcal{B} \in \text{CAT}^0$ and f convex, \mathcal{S} takes on positive curvature of gluing seams! In this case, at each gluing seam attach a “fin”, i.e. a copy of $\{(p, u) : u \geq \epsilon f(p)\}$, to recover CAT^0 (compare billiards, §6.(2)(c)). See figure, where \mathcal{X} is the simply connected cover of the illustrated surface.



(4) *Proving “only if” in characterization theorem:*

- (a) CAT^κ has a rich theory of *subspaces*. (A problematical topic in CBB^κ .)⁷ Say $\mathcal{U} \subset \mathcal{X}$ has *extrinsic curvature* $\leq A$ if intrinsic distance s in \mathcal{U} and extrinsic distance r in \mathcal{X} satisfy $s - r \leq \frac{A^2}{24}r^3 + o(r^3)$ for s sufficiently small. For subspaces $\mathcal{U} = \{p\} \times \mathcal{F}$, apply:
A.-Bishop Gauss-equation theorem [AB 06].
 $(\text{Extrinsic curvature } \mathcal{U}) \leq A \implies \text{curv } \mathcal{U} \leq \kappa + A^2$.
- (b) CBB^κ allows a standard method of proof by induction on $n = \dim \mathcal{X}$: Prove for $n = 1$, then show the theorem passes from direction spaces $\Sigma_p \mathcal{X}$ (dimension $n - 1$) to \mathcal{X} .
- (c) Proof of CBB^κ gluing condition requires Petrunin incomplete-globalization theorem (§14.(2)(a)): Set $\mathcal{B}_0^\dagger = \mathcal{B}^\dagger - (f^\dagger)^{-1}(0)$. Then $(\mathcal{B}_0^\dagger)^c \cong \mathcal{B}^\dagger$. Thus it suffices to prove \mathcal{B}_0^\dagger is geodesic, and $\text{curv } \mathcal{B}_0^\dagger \geq \kappa$.

⁷Celebrated problem: does ∂S inherit a curvature bound if $S = \text{convex set in } \mathcal{X} \in \text{CBB}^\kappa$?

14. GLOBALIZATION II

(1) • Let \mathcal{X} = complete intrinsic space with $\text{curv } \mathcal{X} \leq \kappa$.

(a) *A.-Bishop no-conjugate-point theorem* [AB90]. Let γ be a simple geodesic, $\ell(\gamma) < \pi^\kappa$. Then there is a space $\mathcal{N} \in \text{CAT}^\kappa$, a geodesic $\hat{\gamma}$ in an open subset $\hat{\Omega} \subset \mathcal{N}$, and an open isometric embedding $\Phi : \hat{\Omega} \rightarrow \mathcal{X}$ with $\Phi \circ \hat{\gamma} = \gamma$. (If γ not simple, use isometric immersion.)

(b) Proof by Reshetnyak gluing (§5) of small CAT^κ balls along $\hat{\alpha}$.⁸

(c) Then local geodesics move homotopically as endpoints follow preassigned paths, stopping only if length reaches π^κ . Using Alexandrov patchwork (§3), can now prove Hadamard-Cartan theorem (§3) as well as the following (each leading to the next):

Radial lemma [AB96]. If geodesics from p to points of $B(p, \pi^\kappa/2)$ exist, are unique and vary continuously, then $B(p, \pi^\kappa/2) \in \text{CAT}^\kappa$.

Lifting globalization theorem [AKP]. $\exists \mathcal{B} \in \text{CAT}^\kappa$ with $\mathcal{B} = B(\hat{p}, \pi^\kappa/2)$ for some $\hat{p} \in \mathcal{B}$, and a local isometry $\Phi : \mathcal{B} \rightarrow \mathcal{X}$ with $\Phi(\hat{p}) = p$, such that for any curve $\alpha : [0, 1] \rightarrow \mathcal{X}$ with $\alpha(0) = p$ and $\text{length } \alpha < \pi^\kappa/2$, there is a unique path $\hat{\alpha} : [0, 1] \rightarrow \mathcal{B}$ such that $\hat{\alpha}(0) = \hat{p}$ and $\Phi \circ \hat{\alpha} = \alpha$.

Generalized Hadamard-Cartan theorem [AKP] (due to Bowditch if \mathcal{X} is proper [Bo95]). $\mathcal{X} \in \text{CAT}^\kappa \iff \mathcal{X}$ is $2 \cdot \pi^\kappa$ -simply connected, i.e. any closed curve of length $< 2 \cdot \pi^\kappa$ is null-homotopic through closed curves of length $< 2 \cdot \pi^\kappa$.

(2) (a) *Petrinin incomplete-globalization theorem* [P]. Let X be a geodesic space and X^c be its completion. If $\text{curv } X \geq \kappa$, then $X^c \in \text{CBB}^\kappa$.

⁸The no-conjugate-point theorem of [AB90] applies to a larger class of spaces and required a different proof. This CAT^κ formulation is due to Lytchak.

15. POLYHEDRAL EXAMPLES

• \mathcal{X} = spherical, Euclidean or hyperbolic polyhedral space, i.e. \exists finite triangulation by simplices σ , σ isometric to simplex σ^m in $\mathcal{M}^m(\kappa)$, fixed κ , some m .⁹

- (1) (a) A neighborhood of $p \in \mathcal{X}$ is isometric to a tip of $\text{Cone}_\kappa \Sigma_p$.
 (b) $\text{Link } \sigma = \{\sigma' : \sigma \cap \sigma' = \emptyset; \sigma \text{ and } \sigma' \text{ are faces of the same simplex}\}$.
 (c) $\text{Link } \sigma$ carries natural structure of spherical polyhedron.
 (d) If p = interior point of k -simplex σ , then $T_p \mathcal{X} \cong \mathbf{E}^k \times (\text{Cone Link } \sigma)$,
 so $\Sigma_p \mathcal{X} \cong k$ -th spherical suspension, $(\text{Cone}_1)^k \text{Link } \sigma$.

(2) (a) *Theorem.* $\mathcal{X} \in \text{CBB}^\kappa \iff$ following conditions hold:

- (i) \mathcal{X} is *pure*, i.e. $\exists m \geq 0$ such that any $\sigma =$ face of some σ^m .
- (ii) $\text{Link } \sigma^{m-1} =$ one or two points.
- (iii) $\text{Link } \sigma^k$, $k \leq m - 2$, is connected.
- (iv) $\text{Link } \sigma^{m-2}$ isometric to circle of length $\leq 2 \cdot \pi$ or closed interval of length $\leq \pi$.

(b) *Theorem.* $\text{curv } \mathcal{X} \leq \kappa$

\iff any connected component of any $\text{Link } \sigma$ is $(2 \cdot \pi)$ -simply connected

\iff any closed local geodesic in Link_σ has length at least $2 \cdot \pi$.

⁹Source for this section: [AKP].

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