Gromov Hyperbolic Spaces

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1 Abstact

The class of Gromov hyperbolic spaces form a particularly nice class of metric spaces, where the global asymptotic geometry can be studies by very elementary methods. With asymptotic geometry we mean the study the large scale aspects of metric spaces where we ignore completely the local geometry of the spaces in question. In particular the spaces can be discrete which says in some sense that the local geometry does not exist.

For Gromov hyperbolic spaces one can define a *boundary at infinity* and it turns out that (under some natural assumptions) the whole asymptotic geometry can be decoded by the properties of the boundary at infinity.

A space X is called *Gromov hyperbolic*, if there exists a constant $\delta \geq 0$ such that for any quadruple of points $(x, y, z, w) \in X^4$ the two largest of the following three numbers

$$|xy| + |zw| \,,\; |xz| + |yw| \,,\; |xw| + |yz|$$

differ by at most 2δ. Here $|xy|, |zw|$ etc. denotes the distances. It is surprising that this simple property catches some important properties of the classical hyperbolic space H^n but is flexible enough in order to apply it for a broad class of spaces.

The classical hyperbolic space has an ideal boundary $\partial_\infty \mathrm{H}^n$ which is S^{n-1} in the unit ball and $\mathbb{R}^{n-1} \cup \{\infty\}$ in the upper half space model. In this classical sitation there is a deep and well known connection between the geometry of H^n and the Möbius geometry of its boundary. In particular an isometric map $F : \mathrm{H}^n \to \mathrm{H}^n$ induces a Möbius map $f = \partial_\infty F : \partial_\infty \mathrm{H}^n \to \partial_\infty \mathrm{H}^n$ and vice versa, a Möbius map of the boundaries comes from an isometry of the hyperbolic space.

To a large extend, this interplay between space and boundary can be generalized to to general Gromov hyperbolic spaces.

The main focus of the course is an introduction to the basic concepts of the theory and a more detailed study of the relation between the space and its boundary at infinity.